Locally Roelcke precompact Polish groups

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Group uniformities and Roelcke precompact groups

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To every topological group, G, there are four associated uniformities: the *left*, *right*, *two-sided*, and *Roelcke* uniformities, each compatible with the topology.

In a Polish group, these uniformities are all metrizable and the uniform notions coincide with their metric analogues in a compatible metric.

Group uniformities and Roelcke precompact groups

- The left uniformity is generated by the entourages

$$\{(g,h) \in G^2 \mid g \in hV\}$$

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- The Roelcke uniformity is generated by entourages

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again as V ranges over neighborhoods of 1_G.

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A substantial theory of these groups has been developed in recent years by, e.g., V.V. Uspenskij and T. Tsankov, and there are strong connections between their topological group properties and the properties of structures from which they arise as transformations.

Coarse structures and locally bounded groups

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For example, for a metric space, (X, d), the entourages,

$$\{(x, y) \in X^2 \mid d(x, y) \leqslant r\},$$

as r varies over $\mathbb{R}^+$, generates a coarse structure on X.

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$$\sup\{\rho(g, h) \mid g, h \in A\} < \infty$$

for every continuous, left-invariant pseudometric, ρ, on G.

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We denote the ideal of coarsely bounded sets by $\mathcal{OB}(G)$ (or just $\mathcal{OB}$ when G is implicit).

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Definition

A Polish group, G, is *coarsely bounded* when G is a coarsely bounded subset of itself, i.e., when every compatible left-invariant metric on G has finite diameter.

This means that G has trivial large-scale geometry (it is coarsely equivalent to a point).

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Definition

A Polish group, G, is *locally bounded* when there is a coarsely bounded identity neighborhood $U \subseteq G$.

This means that G has a metrizable large-scale geometry. It admits a *coarsely proper* metric: a left invariant metric assigning infinite diameter to all sets that are not in $\mathcal{OB}$.

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Theorem (Rosendal)

A subset $A \subseteq G$ is coarsely bounded if and only if for every identity neighborhood $V \subseteq G$ there is a finite $F \subseteq G$ and $k \in \mathbb{N}$ so that $A \subseteq (FV)^k$.

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Thus a Polish group G is coarsely bounded if and only if for every V there is a finite F and bound k so that $G = (FV)^k$.

Recalling the definition of Roelcke precompactness for a Polish group, we see this as being a special case of coarse boundedness: for every V there is a finite F so that $G = VFV$.

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Definition

A Polish group, G, is *locally Roelcke precompact* if it possesses a Roelcke precompact identity neighborhood.

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Definition

A metric space, (X, d), *closely embeds amalgams* if, for every finite $A \subseteq X$ and $\varepsilon > 0$, there is a $\delta > 0$ so that if $i, j, k, l : A \hookrightarrow X$ have

$$|d(i(a), j(b)) - d(k(a), l(b))| < \delta \text{ for every } a, b \in A,$$

then there are $k', l' : A \hookrightarrow X$ with $d(k(a), l(b)) = d(k'(a), l'(b))$ and $\max\{d(i(a), k'(a)), d(j(a), l'(a))\} < \varepsilon$ for all $a, b \in A$.

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Theorem

Suppose (X,d) is a separable, complete, ultrahomogeneous metric space closely embedding amalgams. Then if $p \in X$ and $r \in \mathbb{R}^+$,

$$V_{p,r} = \{f \in \mathrm{Iso}(X) \mid d(p,f(p)) < r\}$$

is a Roelcke precompact subset of $\mathrm{Iso}(X)$.

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  - New examples: $\mathrm{Iso}(\mathbb{U})$, the automorphism group of the countably branching tree—more generally, $\mathrm{Aut}(\Gamma)$ for any metrically homogeneous graph, Γ
- (Rosendal) the group of orientation-preserving homeomorphisms of $\mathbb{R}$ that commute with integral shifts (and the corresponding subgroup of $\mathrm{Aut}(\mathbb{Q}, <)$)

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Similarly, every $g \in G$ has a compact neighborhood in $\overline{G}^{\wedge}$. So this question amounts to asking if every $x \in \overline{G}^{\wedge} \setminus G$ has a compact neighborhood.

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Example

Let $G = \mathrm{Aut}(T_\infty)$. For any $a, b \in T_\infty$, $x \in \overline{G}^{\wedge}$, and Roelcke-Cauchy sequence $f_n \to x$, the f_n's eventually agree on the distance $d(a, f_n(b))$.

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If $a, b \in T_\infty$ are fixed and $K \subseteq \overline{G}^{\wedge}$ is compact, then the set $\{r \in \mathbb{R} \mid \exists (f_n) \ \lim f_n \in K \text{ and } \lim d(a, f_n(b)) = r\}$ is finite.

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Then construct $y \in \overline{G}^{\wedge}$ so that $\lim d(a', g_n(b'))$ exceeds this bound for all $a', b' \in T_\infty$ and all $g_n \to y$.

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Then construct $y \in \overline{G}^{\wedge}$ so that $\lim d(a', g_n(b'))$ exceeds this bound for all $a', b' \in T_\infty$ and all $g_n \to y$.

This property will also hold for all $z \in G \cdot y$, and therefore, $G \cdot y \cap K = \emptyset$.

Definition (Rosendal)

A Polish group, G, has *bounded geometry* if there is an $A \in \mathcal{OB}$ so that every $B \in \mathcal{OB}$ is covered by finitely-many translates of A.

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Adapting the proof of this characterization to the specific case of locally Roelcke precompact groups, one sees that such groups have bounded geometry *exactly when the above strategy works*.

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Adapting the proof of this characterization to the specific case of locally Roelcke precompact groups, one sees that such groups have bounded geometry *exactly when the above strategy works*.

Proposition

A locally Roelcke precompact Polish group, G, has bounded geometry if and only if there is a compact neighborhood K of 1_G in $\overline{G}^\wedge$ with $\overline{G}^\wedge = G \cdot K$.

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On the other hand, the Roelcke completion of $\mathrm{Aut}(T_\infty)$ *is* locally compact.

For any $a \in T_\infty$, if $V_a = \{f \in \mathrm{Aut}(T_\infty) \mid f(a) = a\}$ and $g \in \mathrm{Aut}(T_\infty)$, then $V_a g V_a$ is relatively Roelcke precompact. In other words, $\mathrm{Aut}(T_\infty)$ is *uniformly* locally precompact, and so $\overline{\mathrm{Aut}(T_\infty)}^{\wedge}$ is (uniformly) locally compact.

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In general, if U is a relatively Roelcke precompact identity neighborhood in G, then each gU is relatively Roelcke precompact and so it suffices that their product, UgU, remains Roelcke precompact.

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$A \subseteq VFV$

- closed under taking subsets
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- closed under left/right translations
- stable under inversion

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- stable under inversion
- stable under topological closure

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The ideal of relatively Roelcke precompact sets

$\mathcal{R}$

$A \subseteq VFV$

- closed under taking subsets
- closed under finite unions
- closed under left/right translations
- stable under inversion
- stable under topological closure
- not always stable under products

$\mathcal{OB}$

$A \subseteq (FV)^k$

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- closed under finite unions
- closed under left/right translations
- stable under inversion
- stable under topological closure
- stable under products

Example

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Let $G = \mathbb{Z}^{\mathbb{N}} \rtimes S_\infty$. If $A = \{(0, 0, \dots)\} \times S_\infty$, then A is relatively Roelcke precompact. Let $g = ((0, 1, 2, \dots), 1_{S_\infty})$. Then A and gA are in $\mathcal{R}$, while their product AgA is not.

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However, for some classes of groups, this is the case:

Theorem

If a Polish group, G, is Weil-complete (CLI) or locally Roelcke precompact, then $\mathcal{R}$ is stable under products.

A characterization

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Theorem

The following are equivalent for a Polish group, G:

- G *is locally Roelcke precompact*
- $\overline{G}^{\wedge}$ *is locally compact*
- $\overline{G}^{\wedge}$ *is uniformly locally compact*
- G *is locally bounded and* $\mathcal{R}(G) = \mathcal{OB}(G)$

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A Polish group is Roelcke precompact if and only if it is locally Roelcke precompact and coarsely bounded.

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- While recent advances have highlighted the role of Roelcke precompact groups, such groups are trivial from the perspective of large-scale geometry. The locally Roelcke precompact groups are a modest generalization that can admit interesting coarse geometry.

- They furnish another class of groups for which the coarsely bounded sets are more tractable—they are, for locally Roelcke precompact G, precisely the traces of the compact sets in locally compact space in which G is densely embedded.

Thank you!