Some categorical aspects of coarse spaces and balleans

Nicolò Zava
joint work with Dikran Dikranjan

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Contents

1. Basics on the classical theory of metric spaces:
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   - relationship between coarse spaces and balleans.

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    - quasi-isometries and coarse equivalences;
    - finitely generated groups;
2. Beyond metric spaces:
    - coarse spaces;
    - balleans;
    - relationship between coarse spaces and balleans.
3. Coarse category:
    - definition of **Coarse** and **Coarse**$/_\sim$;
    - epimorphisms and monomorphisms in **Coarse**;
    - products, coproducts and quotients in **Coarse**;
    - epimorphisms and monomorphisms in **Coarse**$/_\sim$.

Metric spaces and finitely generated groups

Definition (Coarse equivalence)

Let (X,d) and (Y,d') be two metric spaces. A map $f\colon X\to Y$ is a coarse equivalence if:

1) $f(X)$ is a net in Y (i.e., there exists $\varepsilon\geq 0$ such that $B(f(X),\varepsilon)=Y$);
2) there exist $\rho_-,\rho_+\colon\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ such that $\rho_-,\rho_+\xrightarrow{+\infty}+\infty$ and, for every $x,y\in X$,
$$\rho_-(d(x,y))\leq d'(f(x),f(y))\leq\rho_+(d(x,y)).$$

Two spaces are coarsely equivalent if there exists a coarse equivalence between them.

A quasi-isometry is a coarse equivalence such that ρ_- and ρ_+ are affine.

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A quasi-isometry is a coarse equivalence such that ρ_- and ρ_+ are affine.
Coarse equivalence and quasi-isometry are equivalence relations.

- The inclusion of a net into a metric space is a quasi-isometry.
- $n^2 \mapsto n^3$ is a coarse equivalence between $\{n^2 \mid n \in \mathbb{N}\}$ and $\{n^3 \mid n \in \mathbb{N}\}$, but it is not a quasi-isometry.
- $f: \mathbb{Z} \to \{0\}$ is not a coarse equivalence.

A group G is finitely generated if there exists a finite set $\Sigma \subseteq G$ of generators of G.

Let G be a finitely generated group and $\Sigma = \Sigma^{-1}$ be a finite subset of generators of G. Define the word metric relative to Σ between two points $g, h \in G$ the value

$$d_\Sigma(g,h) = \begin{cases} \min\{n \in \mathbb{N} \mid \exists \sigma_1, \dots, \sigma_n \in \Sigma : \; g^{-1}h = \sigma_1 \cdots \sigma_n\} & \text{if } g \neq h, \\ 0 & \text{otherwise.} \end{cases}$$

d_Σ is invariant under left multiplication (i.e., $d_\Sigma(kg, kh) = d_\Sigma(g,h)$, for every $g, h, k \in G$).

Theorem (Indipendence from the generator set)

Let G be a finitely generated group and Σ and Δ be two symmetric finite generators subsets. Then (G, d_Σ) and (G, d_Δ) are quasi-isometric.

Beyond metric spaces: coarse spaces and balleans

Definition (Roe, 2003)

Let X be a set. A coarse structure $\mathcal{E}$ on X is a subset of $\mathcal{P}(X\times X)$ s.t.:
1) $\Delta_X=\{(x,x)\mid x\in X\}\in\mathcal{E}$;
2) $\mathcal{E}$ is closed under subsets;
3) $\mathcal{E}$ is closed under finite unions;
4) if $E,F\in\mathcal{E}$, then $E\circ F=\{(x,z)\mid\exists y:\,(x,y)\in E,(y,z)\in F\}\in\mathcal{E}$;
5) if $E\in\mathcal{E}$, then $E^{-1}=\{(y,x)\mid(x,y)\in E\}\in\mathcal{E}$.

$(X,\mathcal{E})$ is a coarse space.

Properties (2) and (3) imply that $\mathcal{E}$ is an ideal of subsets of $X\times X$.

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The definition is quite similar to the one of uniformity.

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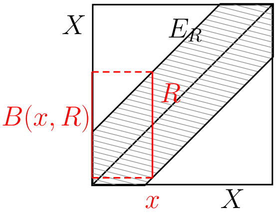
$(X, \mathcal{E})$ is a coarse space.

- $\mathcal{T}_X = \{E \subseteq \Delta_X\}$ is the trivial coarse structure over X.
- $\mathcal{M}_X = \mathcal{P}(X \times X)$ is the indiscrete coarse structure over X.
- If (X, d) is a metric space, the family of all $E \subseteq X \times X$ such that $E \subseteq E_R = \{(x,y) \mid d(x,y) \leq R\}$, for some $R \geq 0$, is the metric coarse structure.



Morphisms

A subset L of a coarse space $(X, \mathcal{E})$ is large in X if exists $E \in \mathcal{E}$ such that $E[L] = \{y \mid (x, y) \in E, x \in L\} = X$.

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A map $f\colon(X,\mathcal{E})\to(Y,\mathcal{F})$ between coarse spaces is:

- bornologous (coarsely uniform) if $(f\times f)(E)=\{(f(x),f(y))\mid (x,y)\in E\}\in\mathcal{E}$, for every $E\in\mathcal{E}$;
- effectively proper if $(f\times f)^{-1}(F)=\{(x,y)\mid (f(x),f(y))\in F\}\in\mathcal{E}$, for every $F\in\mathcal{F}$;
- a coarse embedding if it is bornologous and effectively proper;
- an asymorphism if it is bijective and both f and f^{-1} are bornologous;
- a coarse equivalence if it is a coarse embedding and $f(X)$ is large in Y.

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- a coarse embedding if it is bornologous and effectively proper;
- an asymorphism if it is bijective and both f and f^{-1} are bornologous;
- a coarse equivalence if it is a coarse embedding and $f(X)$ is large in Y.

f is a coarse equivalence if and only if it is bornologous and there exists another bornologous map $g\colon Y\to X$ (called coarse inverse) such that $f\circ g\sim id_Y$ e $g\circ f\sim id_X$.

Definition (Protasov, Banakh, 2003)

A ball structure is a triple $\mathfrak{B} = (X, P, B)$, where X and P are two sets, $P \neq \emptyset$, (called support and radii set of $\mathfrak{B}$, respectively) and $B\colon X \times P \to \mathcal{P}(X)$ is a map that associates a subset $x \in B(x, \alpha)$ of X, called ball centered in x with radius α, to each pair $(x, \alpha) \in X \times P$.

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If (X, P, B) is a ball structure, for every $x \in X$, $\alpha \in P$ and $A \subseteq X$, put

$$B^*(x, \alpha) = \{y \mid x \in B(y, \alpha)\} \quad \text{and} \quad B(A, \alpha) = \bigcup_{x \in A} B(x, \alpha).$$

A ball structure $\mathfrak{B} = (X, P, B)$ is called:

i) upper symmetric if, for every $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha') \quad \text{e} \quad B^*(x, \beta) \subseteq B(x, \beta');$$

ii) upper multiplicative if, for every $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Definition (Protasov, Banakh, 2003)

A ballean is an upper symmetric and upper multiplicative ball structure.

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$$B^*(x,\alpha) = \{y \mid x \in B(x,\alpha)\} \quad \text{and} \quad B(A,\alpha) = \bigcup_{x\in A} B(x,\alpha).$$

A ball structure $\mathfrak{B} = (X, P, B)$ is called:

i) lower symmetric if, for every $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x,\alpha') \subseteq B^*(x,\alpha) \quad \text{e} \quad B^*(x,\beta') \subseteq B(x,\beta);$$

ii) lower multiplicative if, for every $\alpha \in P$, there exists $\beta \in P$ such that, for every $x \in X$,

$$B(B(x,\beta),\beta) \subseteq B(x,\alpha).$$

Lower symmetric and lower multplicative ball structures provide an equivalent description to uniformities.

- $\mathfrak{B}_{\mathcal{T}} = (X, P, B_{\mathcal{T}})$ such that $B_{\mathcal{T}}(x, \alpha) = \{x\}$, for every $x \in X$ and $\alpha \in P$, is the trivial ballean.
- $\mathfrak{B}_{\mathcal{M}} = (X, P, B_{\mathcal{M}})$ such that there exists a radius $\alpha \in P$ such that $B_{\mathcal{M}}(x, \alpha) = X$ is the indiscrete ballean (or bounded ballean).
- If (X, d) is a metric space, then $\mathfrak{B}_d = (X, \mathbb{R}_{\geq}0, B_d)$ is the metric ballean.

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- $\mathfrak{B}_{\mathcal{M}} = (X, P, B_{\mathcal{M}})$ such that there exists a radius $\alpha \in P$ such that $B_{\mathcal{M}}(x, \alpha) = X$ is the indiscrete ballean (or bounded ballean).
- If (X, d) is a metric space, then $\mathfrak{B}_d = (X, \mathbb{R}_{\geq} 0, B_d)$ is the metric ballean.

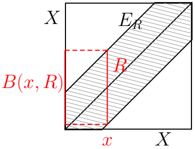
Example (Group ballean)

Let G be a group. A group ideal $\mathcal{I}$ over G is a family of subsets of G which contains a non-empty element, is closed under taking subsets, under finite unions (hence it is an ideal), under product of two elements (i.e., if $F, K \in \mathcal{I}$, then $FK = \{gh \mid g \in F, h \in H\} \in \mathcal{I}$) and under inverse of elements (i.e., if $I \in \mathcal{I}$, then $I^{-1} = \{g^{-1} \mid g \in I\} \in \mathcal{I}$).
$\mathfrak{B}_{\mathcal{I}} = (G, \mathcal{I}, B_{\mathcal{I}})$ is a group ballean, where

$$B_{\mathcal{I}}(g, I) = g(I \cup \{e\})$$

for every $g \in G$ and $I \in \mathcal{I}$.

If (X, d) is a metric space, the family of all $E \subseteq X \times X$ such that $E \subseteq E_R = \bigcup_x \{x\} \times B(x, R)$ for some $R \geq 0$ is a coarse structure.



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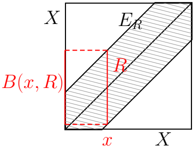

- If $\mathfrak{B} = (X, P, B)$ is a ballean, then the family of all subsets E for which there exists $\alpha \in P$ such that

$$E \subseteq E_\alpha = \bigcup_{x \in X} \{x\} \times B(x, \alpha)$$

  is a coarse structure $\mathcal{E}_\mathfrak{B}$ over X.
- If $(X, \mathcal{E})$ is a coarse space, then $\mathfrak{B}_\mathcal{E} = (X, \mathcal{E}_\Delta, B_\mathcal{E})$, where $\mathcal{E}_\Delta = \{E \mid \Delta_X \subseteq E\}$ and

$$B_\mathcal{E}(x, E) = E[x] = \{y \mid (x, y) \in E\}$$

  for every $x \in X$ and $E \in \mathcal{E}_\Delta$, is a ballean with X as support.

Coarse spaces and balleans are equivalent constructions.

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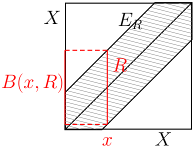

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  for every $x \in X$ and $E \in \mathcal{E}_\Delta$, is a ballean with X as support.

Coarse spaces and balleans are equivalent constructions.

A third way to describe large scale geometry of a space is given by the large scale structures (Dydak, Hoffland, 2008), also named asymptotic proximities (Protasov, 2008).

Let L be a subset of a ballean (X, P, B). Then L is large in X if and only if there exists $\alpha \in P$ such that $B(L, \alpha) = X$.

Let $f, g\colon S \to X$ be two maps from a non-empty set to a ballean (X, P, B). $f \sim g$ if and only if there exists $\alpha \in P$ such that $f(x) \in B(g(x), \alpha)$, for every $x \in X$.

If $f\colon (X, P_X, B_X) \to (Y, P_Y, B_Y)$ is a map between balleans, then:

1) f is bornologous if and only if, for every $\alpha \in P_X$, there exists $\beta \in P_Y$ such that $f(B_X(x, \alpha)) \subseteq B_Y(f(x), \beta)$, for every $x \in X$;
2) f is effectively proper if and only if, for every $\alpha \in P_Y$, there exists $\beta \in P_X$ such that $f^{-1}(B_Y(f(x), \alpha)) \subseteq B_X(x, \beta)$, for every $x \in X$.

Coarse categories

We consider two coarse categories.

- The category **Coarse** has coarse spaces as objects and bornologous maps between them as morphisms: $\mathrm{Mor}_{\mathbf{Coarse}}(X, Y)$, where X and Y are coarse spaces, is the family of all bornologous maps $f\colon X \to Y$.

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- The category **Coarse** has coarse spaces as objects and bornologous maps between them as morphisms: $\mathrm{Mor}_{\mathbf{Coarse}}(X, Y)$, where X and Y are coarse spaces, is the family of all bornologous maps $f\colon X \to Y$.
- If X and Y are coarse spaces, closeness $\sim$ is a congruence (i.e., if $f, g\colon X \to Y$ and $h, k\colon Y \to Z$ are maps between coarse spaces such that $f \sim g$ and $h \sim k$, then $h \circ f \sim k \circ g$). Define the quotient category **Coarse**$/_\sim$ whose objects are coarse spaces and morphisms are the families

$$\mathrm{Mor}_{\mathbf{Coarse}/_\sim}(X, Y) = \mathrm{Mor}_{\mathbf{Coarse}}(X, Y)/_\sim,$$

where X and Y are coarse spaces.

A morphism $\alpha\colon X\to X'$ of a category $\mathcal{X}$ is called:

- an epimorphism if, for every pair of morphisms $\beta,\gamma\colon X'\to X''$, $\beta=\gamma$ whenever $\beta\circ\alpha=\gamma\circ\alpha$ (i.e., α is right-cancellative);
- a monomorphism if, for every pair of morphisms $\beta,\gamma\colon X''\to X$, $\beta=\gamma$ whenever $\alpha\circ\beta=\alpha\circ\gamma$ (i.e., α is left-cancellative);
- a bimorphism if it is both an epimorphism and a monomorphism;
- an isomorphism if there exists a morphism $\beta\colon X'\to X$, called inverse of α, such that $\alpha\circ\beta=1_X$ and $\beta\circ\alpha=1_{X'}$.

Every isomorphism is a bimorphism, but the opposite implication does not hold in general. If it happens, the category is called balanced.

The isomorphisms of **Coarse** are precisely the asymorphisms.

Theorem

The category **Coarse** *is topological (in the sense of Herrlich).*

Some consequences.

- The epimorphisms of **Coarse** are the surjective morphisms.
- The monomorphisms of **Coarse** are the injective morphisms.
- The category **Coarse** is not balanced: if X has at least two points, then the identity $f\colon (X, \mathcal{T}_X) \to (X, \mathcal{M}_X)$ is a bimorphism, but it is not an isomorphism.

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Since the family of all the coarse structure $\mathfrak{C}(X)$ over a set X is a complete lattice,

- arbitrary products,
- arbitrary coproducts and
- quotients

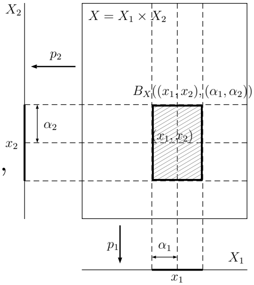
exist.

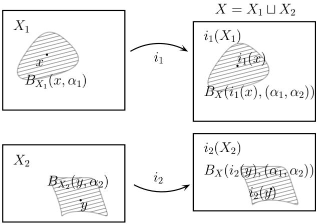
Fix a family $\{\mathfrak{B}_i = (X_i, P_i, B_i)\}_{i\in I}$ of balleans.

Let $X = \prod_i X_i$ and $p_j \colon \prod_i X_i \to X_j$, where $j \in I$, be the projections. Define the product ballean $\prod_i \mathfrak{B}_i = (X, \prod_i P_i, B_X)$, where

$$B_X((x_i)_i, (\alpha_i)_i) = \bigcap_{i\in I} p_i^{-1}(B_i(x_i, \alpha_i)) = \prod_{i\in I} B_i(x_i, \alpha_i),$$

for every $(x_i)_i \in \prod_i X_i$ and $(\alpha_i)_i \in \prod_i P_i$.





Let $X = \bigsqcup_\nu X_\nu$ and $i_\nu \colon X_\nu \to \bigsqcup_\nu X_\nu$, con $\nu \in I$, be the canonical inclusions. Define the coproduct ballean $\coprod_\nu \mathfrak{B}_\nu = (X, \prod_\nu P_\nu, B_X)$, such that

$$B_X(i_\mu(x), (\alpha_\nu)_\nu) = i_\mu(B_\mu(x, \alpha_\mu)),$$

for every $i_\mu(x) \in \bigsqcup_\nu X_\nu$ and $(\alpha_\nu)_\nu \in \prod_\nu P_\nu$.

Quotients of coarse spaces

Fix a coarse space $(X, \mathcal{E})$ and a surjective map $q\colon X \to Y$. Let (X, P, B) be the equivalent ballean.

The quotient structure $\widetilde{\mathcal{E}}^q$ over Y exists, but it is often hard to describe.

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The ball structure $\mathfrak{B}_{\overline{\mathcal{E}}^q}$ is equal to the quotient ball structure $\overline{\mathfrak{B}}^q = (Y, P, \overline{B}^q)$, where

$$\overline{B}^q(y, \alpha) = q(B(q^{-1}(y), \alpha),$$

for every $y \in Y$ and $\alpha \in P$.

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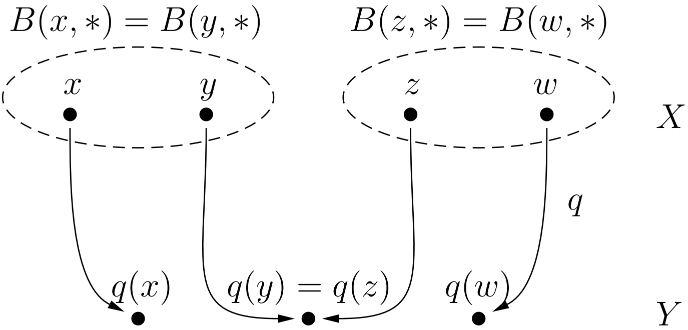
for every $y \in Y$ and $\alpha \in P$.

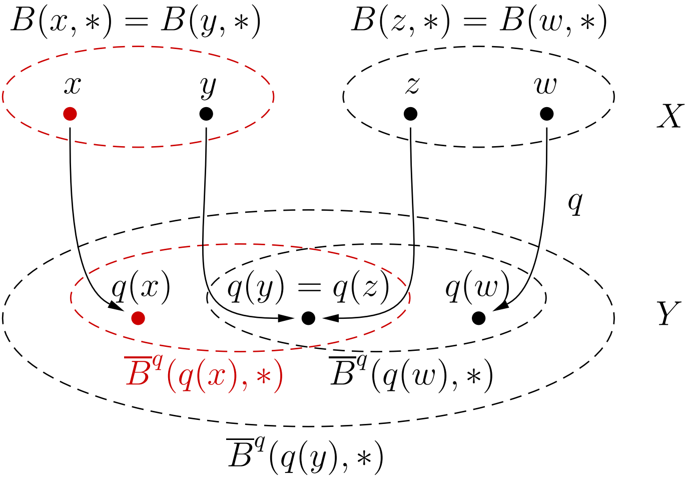
Problem

When $\overline{\mathcal{E}}^q$ is a coarse structure?
Equivalently, when $\overline{\mathfrak{B}}^q = (Y, P, \overline{B}^q)$ is a ballean?

Consider the ballean $\mathfrak{B} = (X, \{*\}, B)$ and the map $q\colon X \to Y$ as described in the following diagram.



Let us now describe which are the balls of the ball structure $\overline{\mathfrak{B}}^q$.



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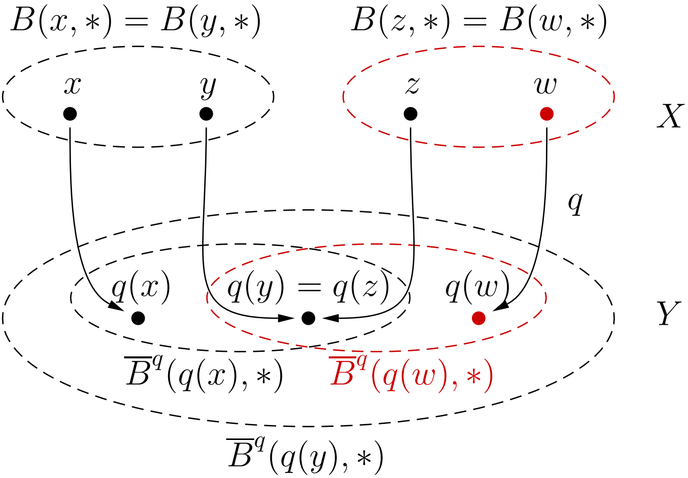

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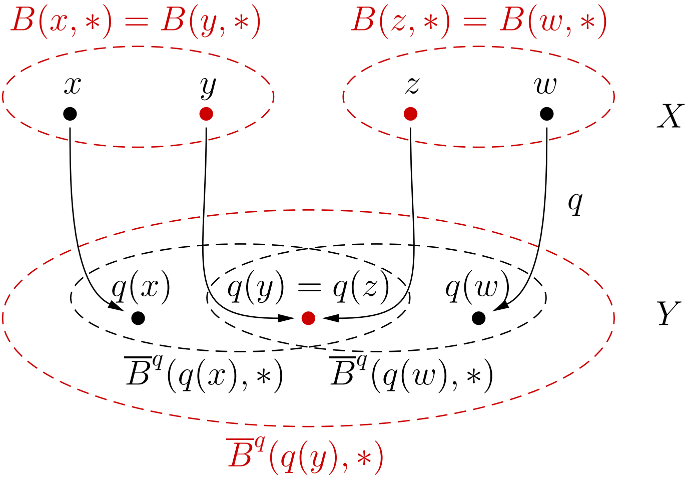

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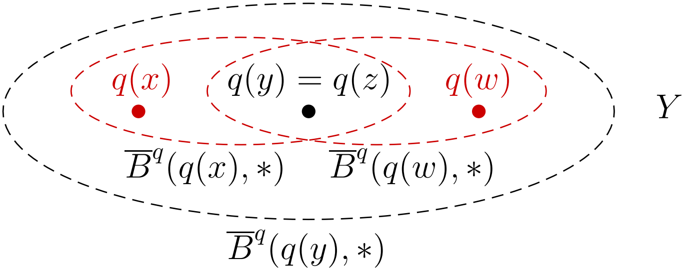

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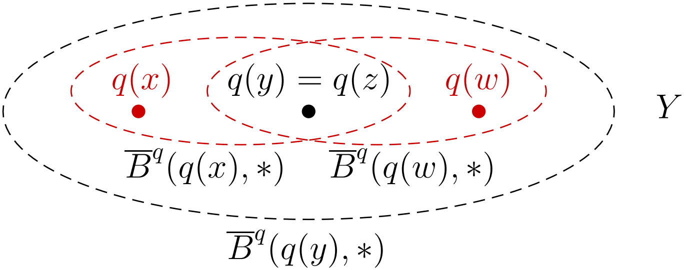

Hence $\overline{\mathfrak{B}}^q$ is not upper multiplicative and so, in particular, it is not a ballean.

Every quotient ball structure is upper symmetric. It eventually fails in being upper multiplicative.

If $q\colon X \to Y$ is a map, $R_q = \{(x, y) \in X \times X \mid q(x) = q(y)\}$.

Definition

Let $(X, \mathcal{E})$ be a coarse space and $q\colon X \to Y$ be a surjective map. Then q is weakly soft if, for every $E \in \mathcal{E}$, there exists $F \in \mathcal{E}$ such that $E \circ R_q \circ E \subseteq R_q \circ F \circ R_q$.

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Theorem

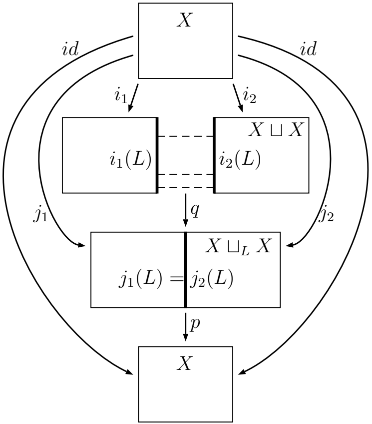
Let $(X,\mathcal{E})$ be a coarse space and $q\colon X\to Y$ be a surjective map. Then q is weakly soft if and only if $\overline{\mathcal{E}}^q$ is a coarse structure.

If $\mathfrak{B}_\mathcal{I}=(G,\mathcal{I},B_\mathcal{I})$ is a group ballean and $q\colon G\to H$ is a quotient homomorphism, then q is weakly soft (if we consider the coarse space $(G,\mathcal{E}_{\mathfrak{B}_\mathcal{I}})$). The quotient ballean $\overline{\mathfrak{B}}^q$ is equivalent to $\mathfrak{B}_{q(\mathcal{I})}=(H,q(\mathcal{I}),B_{q(\mathcal{I})})$, where $q(\mathcal{I})=\{q(K)\mid K\in\mathcal{I}\}$.

Let $\mathfrak{B} = (X, P, B)$ be a ballean and $L \subseteq X$. Define the adjunction space in the following way:
$X \sqcup_L X = X \sqcup X/_{\sim_L}$, where

$$i_\nu(x) \sim_L i_\mu(y) \Leftrightarrow \begin{cases} x = y \in L, \\ \nu = \mu,\ x = y. \end{cases}$$

The map q is weakly soft only in very peculiar cases.



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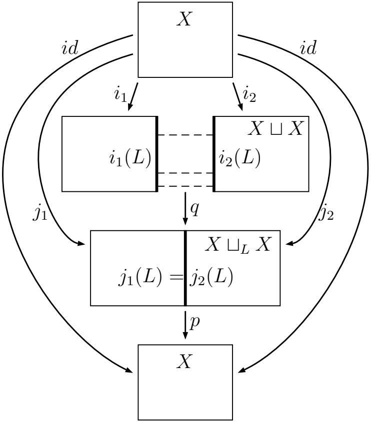

$\mathfrak{B}^a_{X\sqcup_L X} = (X \sqcup_L X, P, B_{X\sqcup_L X})$ is the quotient ballean of $q\colon X \sqcup X \to X \sqcup_L X$, where, for every $x \in X$, $\nu = 1, 2$ and $\alpha \in P$,

$$B_{X\sqcup_L X}(i_\nu(x), \alpha) = \begin{cases} j_\nu(B(x, \alpha)) & \text{if } B(x, \alpha) \cap L = \emptyset, \\ j_1(B(x, \alpha)) \cup j_2(B(x, \alpha)) & \text{otherwise.} \end{cases}$$

Theorem (Epimorphisms)

Let X be a coarse space and $L \subseteq X$. The following are equivalent:
1) L is large X;
2) if $f, g\colon X \to Y$ are bornologous and $f\restriction_L \sim g\restriction_L$, then $f \sim g$;
3) if $f, g\colon X \to Y$ are bornologous and $f\restriction_L = g\restriction_L$, then $f \sim g$.
The equivalence class $[f]_\sim$ of a morphism $f\colon X \to Y$ of **Coarse** *is an epimorphism of* **Coarse**$/_\sim$ *if and only if $f(X)$ is large in Y.*

Theorem (Monomorphisms)

Let $h\colon X \to Y$ be a bornologous map between coarse spaces. T.f.a.e.:
1) h is a coarse embedding;
2) for every coarse space Z and every pair of bornologous maps $f, g\colon Z \to X$, $f \sim g$, whenever $h \circ f \sim h \circ g$.
The equivalence classe $[f]_\sim$ of a morphism f of **Coarse** *is a monomorphism of* **Coarse**$/_\sim$ *if and only if f is a coarse embedding.*

Corollary

The category **Coarse**$/_\sim$ *is balanced.*

Thanks for your attention