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## Recurrence and Universal Minimal Recurrence

(B. Weiss)

$X$  - compact metric space,  $T: X \rightarrow X$  cont.

$x \in X$  is recurrent if  $T^{n_i} x \rightarrow x$ ,  $n_i \rightarrow \infty$ .

More refined notions

1) uniform recurrence :  $\{n : T^n x \in U_x\}$

is syndetic for all neighborhoods  $U_x$  of  $x$ . (syndetic = bounded gaps).

This is equivalent to the orbit closure of  $x$  being a minimal set.

Def:  $x$  is proximal to  $y$  if for some

$$n_i \rightarrow \infty \quad d(T^{n_i} x, T^{n_i} y) \rightarrow 0.$$

$x$  is a distal point if it is not proximal to any  $y \neq x$ .

Thm: Given a system  $(X, T)$  and  $x \in X$  there exists a uniformly recurrent point  $y$  that is proximal to  $x$ .

(related to Hindman's thm that if  $A \subset \mathbb{N}$  contains an IP-set (=finite sums of an infinite sequence  $\{p_j\}$ , i.e.  $\sum_{j \in F} p_j$ ,  $F$  finite subset  $\mathbb{N}$ ) and  $A = A_1 \cup A_2$  then either  $A_1$  or  $A_2$  contains an IP-set).

Def: A subset  $A \subset \mathbb{N}$  is called a central set if for some pair  $(x, y)$  in  $(X, T)$   $x$  is proximal to  $y$ ,  $y$  is uniformly recurrent and  $A = \{n : T^n x \in V_y\}$  where  $V_y$  is ~~a~~ <sup>some</sup> nbd of  $y$ .

$\mathcal{C}$  - central sets.  $\mathcal{C}^*$  - sets that intersect every  $A \in \mathcal{C}$ .

$\mathcal{E}^*$  has the finite intersection property,  
and hence can speak of  $\mathcal{E}^*$ -convergence and recurrence:

A point  $x$  is  $\mathcal{E}^*$ -recurrent if  $\{n: T^n x \in U_x\} \in \mathcal{E}^*$   
for any nbd.  $U_x$  of  $x$ .

Thm: (Furstenberg-W.) The following are equivalent:

- (1)  $x$  is  $\mathcal{E}^*$ -recurrent
  - (2)  $x$  is a distal point in its orbit closure
  - (3) For any system  $(Z, S)$  and recurrent point  $z \in Z$  the point  $(x, z)$  is recurrent in  $(X \times Z, T \times S)$ .
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Erdős and Stone (1945) studied recurrence sets (not quite in this generality),

$A \subset \mathbb{N}$  is a set of recurrence if for every

(minimal) system  $(Z, S)$  there is a point  $z \in Z$   
 $\{n: T^n z \in V_z\} \cap A$  is infinite for all nbds  $V_z$  of  $z$ .

