Prague 25.7.16.

# Recurrence and Universal Minimal Recurrence

(B. Weiss)

$X$ - compact metric space, $T: X \to X$ cont.

$x \in X$ is recurrent if $T^{n_i} x \to x$, $n_i \to \infty$.

More refined notions

1) uniform recurrence : $\{n : T^n x \in U_x\}$ is syndetic for all neighborhoods $U_x$ of $x$. (syndetic = bounded gaps).

This is equivalent to the orbit closure of $x$ being a minimal set.

Def: $x$ is proximal to $y$ if for some $n_i \to \infty$ $d(T^{n_i} x, T^{n_i} y) \to 0$.

$x$ is a distal point if it is not proximal to any $y \neq x$.

Thm: Given a system $(X,T)$ and $x \in X$ there exists a uniformly recurrent point $y$ that is proximal to $x$.

(related to Hindman's thm that if $A \subset \mathbb{N}$ contains an IP-set (= finite sums of an infinite sequence $\{p_j\}$, i.e. $\sum_{j \in F} p_j$, $F$ finite subset $\mathbb{N}$) and $A = A_1 \cup A_2$ then either $A_1$ or $A_2$ contains an IP-set).

Def: A subset $A \subset \mathbb{N}$ is called a central set if for some pair $(x,y)$ in $(X,T)$ $x$ is proximal to $y$, $y$ is uniformly recurrent and $A = \{n : T^n x \in V_y\}$ where $V_y$ is some nbd of $y$.

$\mathcal{C}$ - central sets. $\mathcal{C}^*$ - sets that intersect every $A \in \mathcal{C}$.

$\mathcal{C}^*$ has the finite intersection property, and hence can speak of $\mathcal{C}^*$-convergence and recurrence:

A point $x$ is $\mathcal{C}^*$-recurrent if $\{n : T^n x \in U_x\} \in \mathcal{C}^*$ for any nbd. $U_x$ of $x$.

<u>Thm</u>: (Furstenberg-W.) The following are equivalent:

(1) $x$ is $\mathcal{C}^*$-recurrent

(2) $x$ is a distal point in its orbit closure

(3) For any system $(Z, S)$ and recurrent point $z \in Z$ the point $(x, z)$ is recurrent in $(X \times Z, T \times S)$.

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Erdös and Stone (1945) studied recurrence sets (not quite in this generality).

$A \subset \mathbb{N}$ is a <u>set of recurrence</u> if for every (minimal) system $(Z, S)$ there is a point $z \in Z$ $\{n : T^n z \in V_z\} \cap A$ is infinite for all nbds $V_z$ of $z$.