Metrizable Cantor cubes that fail to be compact in some models for ZF

Eliza Wajch

Siedlce University of Natural Sciences and Humanities
Siedlce, Poland

Toposym, Prague
July 25–29, 2016

Basic notation and concepts

ZF– the Zermelo-Fraenkel system of axioms.

CC(*fin*)– the product of an arbitrary non-empty countable family of non-empty finite sets is non-empty.

$2^J = \{0,1\}^J$–Cantor cube.

CB(X)– the compact bornology of a topological space X (i.e. the collection of all subsets of compact sets in X).

Definition

The bornology **CB**(X) is quasi-metrizable if X admits a compatible quasi-metric d such that **CB**(X) is the collection of all d-bounded sets where a set $A \subseteq X$ is called d-bounded if $A = \emptyset$ or there exist $x \in X$ and a real number $r > 0$ such that

$$A \subseteq B_d(x,r) = \{y \in X : d(x,y) < r\}.$$

The main theorem

The following conditions are all equivalent in every model for **ZF**:

(i) There exist metrizable Cantor cubes that are non-compact.

(ii) There exists a metrizable Cantor cube such that its compact bornology is not quasi-metrizable.

(iii) **CC**(*fin*) fails.

Corollary

In Cohen's Second Model (model M7 in: P. Howard and J. E. Rubin, "Consequences of the Axiom of Choice", Math. Surveys and Monographs 59, Amer. Math. Soc., Providence (RI) 1998), there exist metrizable Cantor cubes that are non-compact. The compact bornologies of such Cantor cubes are not quasi-metrizable.

A basic (quasi)-metrization theorem for products

Notation: If ρ is a (quasi)-metric on X, then $\tau(\rho)$ is the topology on X induced by ρ.

Theorem (Wajch, 2015)

Suppose that J is a countable union of non-empty finite sets. Let $\{(X_j, \tau_j) : j \in J\}$ be a collection of (quasi)-metrizable spaces. Suppose that there exists a collection $\{d_j : j \in J\}$ of (quasi)-metrics such that $\tau_j = \tau(d_j)$ for each $j \in J$. Then it holds true in **ZF** *that the product $\prod_{j\in J}(X_j, \tau_j)$ is (quasi)-metrizable.*

Corollary (Wajch, 2015)

It holds true in **ZF** *that the Cantor cube 2^J is metrizable if and only if J is a countable union of finite sets.*

Non-compact metrizable Cantor cubes

Assumption: $(X_n)_{n\in\omega}$ is a sequence of non-empty finite discrete spaces.

Theorem (A)

If either
(A.1) $\prod_{n\in\omega} X_n$ *is non-compact*
or
(A.2) $\prod_{n\in\omega} X_n = \emptyset$,
then the Cantor cube $2^{\bigcup_{n\in\omega}(X_n\times\{n\})}$ *is non-compact.*

Proof to Theorem (A)

(A.1) Suppose that $\prod_{n\in\omega} X_n$ is non-compact. For each $n \in \omega$ and each $x \in X_n$, let $f_n : X_n \to 2^{X_n}$ be defined by: $[f_n(x)](y) = 1$ if $x = y$, while $[f_n(x)](y) = 0$ if $y \in X_n \setminus \{x\}$. Let

$$f = \prod_{n\in\omega} f_n : \prod_{n\in\omega} X_n \to \prod_{n\in\omega} 2^{X_n}.$$

Then f is a homeomorphic embedding and $Y = f(\prod_{n\in\omega} X_n)$ is closed in $\prod_{n\in\omega} 2^{X_n}$. Now, it suffices to notice that $\prod_{n\in\omega} 2^{X_n}$ and $2^{\bigcup_{n\in\omega}(X_n\times\{n\})}$ are homeomorphic.

(A.2) Suppose that $\prod_{n\in\omega} X_n = \emptyset$. Take an element $\infty \notin \bigcup_{n\in\omega} X_n$ and, for each $n \in \omega$, put $Y_n = X_n \cup \{\infty\}$ with its discrete topology. Then $\prod_{n\in\omega} Y_n$ is non-compact. It follows from (A.1) that $2^{\bigcup_{n\in\omega}(Y_n\times\{n\})}$ is non-compact. Notice that $2^{\bigcup_{n\in\omega}(Y_n\times\{n\})}$ is homeomorphic with $[2^{\bigcup_{n\in\omega}(X_n\times\{n\})}] \times 2^\omega$. Knowing that 2^ω is compact and that finite products of compact spaces are compact, we deduce that $2^{\bigcup_{n\in\omega}(X_n\times\{n\})}$ is non-compact.

Compact bornologies of metrizable Cantor cubes

Assumption: Let J be an uncountable set which is a countable union of pairwise disjont finite sets.

Theorem (B)

The Cantor cube 2^J *is both metrizable and non-compact, while the bornology* $\mathbf{CB}(2^J)$ *is not quasi-metrizable.*

Proof.

I have already shown that 2^J is both metrizable and non-compact. Suppose that $\mathbf{CB}(2^J)$ is quasi-metrizable. This, together with a theorem of Piękosz and Wajch, published in our co-authored article *"Quasi-metrizability of bornological biuniverses in ZF"* (J. Convex Analysis 22 (2015)), implies that there exists a non-empty open set $V \in \mathbf{CB}(2^J)$. There is a finite set $K \subseteq J$ such that $2^{J\setminus K}$ is homeomorphic with a compact subspace of V. By Theorem (A), the Cantor cube $2^{J\setminus K}$ is non-compact. The contradiction obtained shows that $\mathbf{CB}(2^J)$ cannot be quasi-metrizable. □

An additional remark on not second-countable metrizable Cantor cubes

Let k be a fixed positive integer. Suppose that $(X_n)_{n\in\omega}$ is a sequence of non-empty sets such that the set $J = \bigcup_{n\in\omega} X_n$ is uncountable and each X_n has at most k elements. Then 2^J is non-compact. Of course, 2^J is metrizable and not second-countable.

Corollary

Let $k \in \omega \setminus 1$. It holds true in **ZF** *that, up to a homeomorphism, 2^ω is the unique compact Cantor cube 2^J such that J is an infinite set which is a countable union of sets such that each of the sets has at most k-elements.*

Questions

(I) If, in a model **M** for **ZF**, a set J is both uncountable and a countable union of finite sets, must the Cantor cube 2^J be non-compact in **M**?

(II) Is it possible to find easily a model **M** for **ZF** such that **CC**(n) holds in **M** for each $n \in \omega \setminus 1$ and, simultaneously, **CC**(fin) fails in **M**?

Thank you for your attention very much!

Special thanks to Kyriakos Keremedis for his helpful discussion with me through ResearchGate.