Companions of directed sets

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Twelfth Prague Topology Symposium, 25-29 July 2016

Motivation

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This would solve a well known problem first raised in 1968, and would be a major accomplishment:

Is every normal, linearly Lindelöf space Lindelöf?

linearly Lindelöf spaces

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There exists completely regular linearly Lindelöf not Lindelöf spaces. Thus the question raised in 1968:

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linearly Lindelöf Problem

The problem is one of 17 problems discussed by Mary Ellen Rudin in her article "Some Conjectures," in Open Problems in Topology, J. van Mill and G.M. Reed, eds., Elsevier, North-Holland 1990, 184 -193.

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Rudin Conjecture: There is a counterexample, i.e., There exists a normal linearly Lindleöf space that is not Lindelöf.

Sconyers -Howes Claim: There is *no* counterexample, i.e., Every normal linearly Lindleöf space is Lindelöf.

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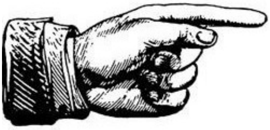

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We review the definitions.

Recall Basic definitions: partial order, linear order, well order

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partial ordered set : if $\leq$ satisfies the *transitive* property: $x \leq y$ and $y \leq z$ imply $x \leq z$.

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Recall Basic definitions, nets and transfinite sequences

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directed set: If $\leq$ is a partial order and every finite set of elements has an upper bound (i.e., for $x, y \in D$ there exists $z \in D$ such that $x, y \leq z$)

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In this terminology, ordinary sequences $f : \omega \to X$ are (transfinite) sequences (where ω denotes the set of natural numbers).

The Ordering Lemma

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Lemma (Ordering Lemma)

For any partially order set $(D, \leq)$ there exists a cofinal $C \subset D$ and a well-order $\preceq$ on C such that $\preceq$ is compatible with $\leq$ *in the sense that if $c_0, c_1 \in C$ and $c_0 \leq c_1$, then $c_0 \preceq c_1$.*

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C is *cofinal* in D means for every $d \in D$ there exists $c \in C$ such that $d \leq c$.

The Ordering Lemma and Companions

Definition

Let $(D, \leq)$ be a partially ordered set, and $(C, \preceq)$ a well ordered set. We say that $(C, \preceq)$ is a *companion of $(D, \leq)$* provided $C \subset D$ is cofinal in $(D, \leq)$, and the well order $\preceq$ on C is compatible with the partial order $\leq$ on C:

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With this definition the Ordering Lemma can be stated simply as

Ordering Lemma: Every partially ordered set has a companion.

Converging and clustering of nets

We recall the well known theory of convergence of J. L. Kelley.

Let $(X, \mathcal{T})$ be a topological space. A net $f : (D, \leq) \to X$ is said to *converge to a point* $x \in X$ provided for every neighborhood U of x, there exists $d \in D$ such that $f(d') \in U$ for all $d' \geq d$. In other words,

$$\uparrow d = \{d' \in D : d' \geq d\} \subset f^{-1}(U)$$

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A net f is said to *cluster at* $x \in X$ (or x *is a cluster point of* f) provided for every neighborhood U of x in X and for every $d \in D$ there exists $d' \geq d$ such that $f(d') \in U$, (in other words, $f^{-1}(U)$ is cofinal in $(D, \leq)$).

Main Topic

Given a net $f : (D, \leq) \to X$, and a companion $(C, \preceq)$ of $(D, \leq)$, there is the automatically the associated a transfinite sequence

$$f \restriction C : (C, \preceq) \to X$$

We call such a transfinite sequence a *companion (transfinite) sequence associated with the net f.*

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QUESTION: What is the relation between convergenc (respectively cluster) of a (companion) transfinite sequence $f \upharpoonright C : (C \preceq) \to X$ and convergent (respectively cluster) of the given net f?

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This question has two interesting positive results (one of which is due to Howes):

Lemma

If either the net f or the companion transfinite sequence $f \restriction C$ converges to a point x, then the other one clusters at x.

Examples show that there are no other implications in general.

Main Topic

In particular, it is possible for a companion sequence $f \restriction C$ to have a cluster point but the net f to have no cluster point.

Converging versus clustering of transfinite sequences

Translating from the previous definitions:

A transfinite sequence $f : \kappa \to X$ converges to $x \in X$ means: for every neighborhood U of x, $f^{-1}(U)$ is final segment of κ.

Converging versus clustering of transfinite sequences

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A transfinite sequence $f : \kappa \to X$ converges to $x \in X$ means: for every neighborhood U of x, $f^{-1}(U)$ is final segment of κ.

A transfinite sequence $f : \kappa \to X$ clusters to $x \in X$ means: for every neighborhood U of x, $f^{-1}(U)$ is a cofinal subset of κ (unbounded in κ).

Garrett Birkhoff (1911-1996)

As is well known for a space X, a set $A \subset X$ and a point $p \in \mathrm{cl}(A) \setminus A$, there is a net $f : D \to A$ into A that converges to x.

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However

Garrett Birkhoff in a paper in 1937 in the Annals of Mathematics gave an example of a space X a set $A \subset X$ and a point $p \in \mathrm{cl}(A) \setminus A$ such that no transfinite sequence in A converges to p. (An easier example can be constructed usng the Tychonoff plank.)

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It seem possible that this pronouncement from such a well know mathematician discouraged further research on transfinite sequences. Birkhoff's statement is correct for convergence of transfinite sequences but not correct regarding *clustering* of transfinite sequences, because "unlimited use of transfinite sequences" would include clustering of transfinite sequences.

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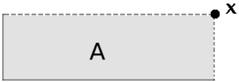
In any case, during the next 25 years there were essentially no publications dealing with the theory of convergence of transfinite sequences.

Garrett Birkhoff vs Howes



Theorem (Howes)

If $x \in \mathrm{cl}(A) \setminus A$ then there exist a transfinite sequence in A that clusters at x.

Proof. From the usual (Kelley) theory of convergence, there is a net $f : (D, \leq) \to X$ such that f map into A and converges to x.

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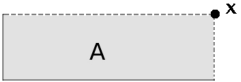

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Proof. From the usual (Kelley) theory of convergence, there is a net $f : (D, \leq) \to X$ such that f map into A and converges to x. By the Ordering Lemma, there is a companion $(C, \preceq)$ of $(D, \leq)$.

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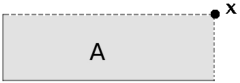

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So let us prove mentioned result.

Garrett Birkhoff vs Howes

We prove: if a net $f : (D, \leq) \to X$ converges to x in X and $(C, \preceq)$ is a companion of $(D, \leq)$ then the companion sequence $f \restriction C$ clusters at x.

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It follows that $(f \restriction C)^{-1}(U)$ is cofinal in $(C, \preceq)$ because otherwise $(f \restriction C)^{-1}(U)$ is bounded in $(C, \preceq)$; say $(f \restriction C)^{-1}(U) \subset [0, c_0]$ where $[0, c_0]$ denotes an initial segment in $(C, \preceq)$.

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But by "directed set" there exists $c_1 \in C$ such that $c_1 > d, c_0$; so $c_0 < c_1$ implies (by compatibility of the orders) that $c_0 \prec c_1$, hence $c_1 \notin [0, c_0)$; hence $c_1 \notin f^{-1}(U)$ which contradicts that $c_1 \geq d$. Contradiction. Thus the companion sequence $(f \restriction C)$ has x as a cluster point. $\square$

Main Question

If a companion transfinite sequence $f \upharpoonright C$ has a cluster point, does the net f have a cluster point?

The answer is "NO" in general, and this is the gap in the "proof" by Sconyers and Howes.

Example

Let λ be an infinite cardinal number and put $(D, \leq) = (\lambda \times \lambda, \leq)$ where $\leq$ is the product order on $\lambda \times \lambda$: $(\alpha, \beta) \leq (\xi, \mu)$ iff $\alpha \leq \xi$ and $\beta \leq \mu$.

Let $\preceq$ denote the lexicographic order on $\lambda \times \lambda$ (i.e., $(\alpha, \beta) <_{lex} (\gamma, \delta)$ iff $\alpha < \gamma$ or $\alpha = \gamma$ and $\beta < \delta$). It is known (in other terminology) that $(D, \preceq)$ is a companion of $(D, \leq)$. This is an example where $C = D$.

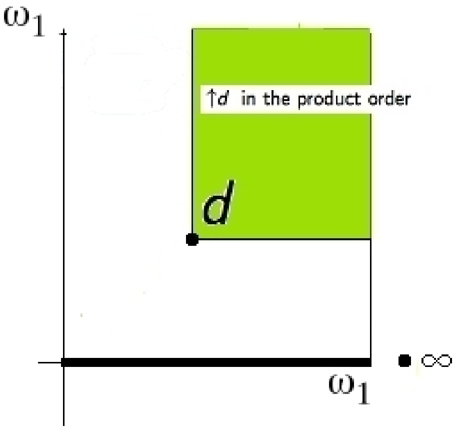
Example of the Gap

On the set $X = (\lambda \times \lambda) \cup \{\infty\}$, define a topology in which all the points in $\lambda \times \lambda$ are isolated and neighborhoods of ∞ have the form

$$U_\alpha = \{(\beta, 0) : \alpha < \beta < \lambda\} \cup \{\infty\}$$

Define a net $f : D \to X$ by $f(d) = d$ for all $d \in D$. Then $f \restriction C = f$ clusters at ∞ since the set $\lambda \times \{0\}$ is cofinal in the lexicographic order, but f has no cluster point since $\lambda \times \{0\}$ is not cofinal in the product order. This completes the proof of the Example.

Example of the Gap



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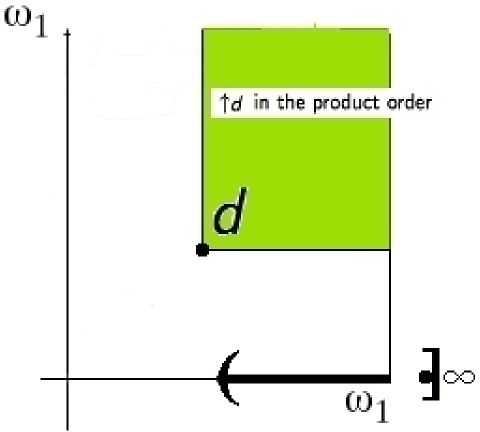

Basic neighborhood of ∞ in the space $X = D \cup \{\infty\}$

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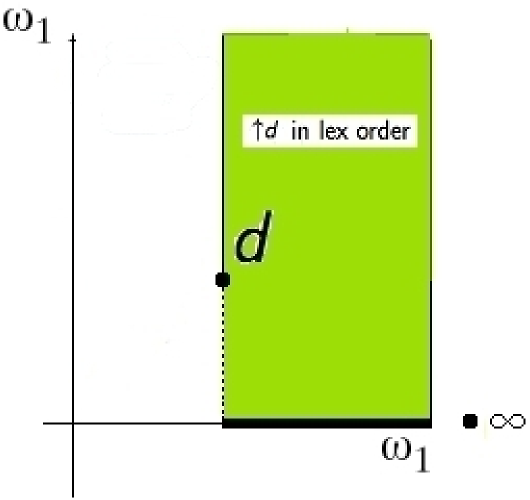

Example of the Gap

We note that $(D, \leq)$ has another companion, the well ordered subset $C' = \{(\alpha, \alpha) : \alpha < \lambda\}$ with $\preceq$ the restriction of $\leq$ to C'. Then for any net $f : D \to X$, because $\preceq$ is $\leq$, $f \restriction C'$ is a subnet of f, hence if $f \restriction C'$ clusters in X, then also f clusters in X.

Example of the Gap

Thus different choices of companion of a directed set $(D, \leq)$ can give different answers to the question:

For a net $f : (D, \leq) \to X$, if $f \restriction C$ clusters at x , does also f cluster at x?

Summary Theorem

Theorem

(1) If $(D, \leq)$ has a well ordered cofinal subset C then use C as the companion and the partial order $\leq$ restricted to C as the well order, and get that $f \restriction C$ is a subnet of f, hence, if $f \restriction C$ clusters at $x \in X$, then the net f clusters at x.

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(1) If $(D, \leq)$ has a well ordered cofinal subset C then use C as the companion and the partial order $\leq$ restricted to C as the well order, and get that $f \upharpoonright C$ is a subnet of f, hence, if $f \upharpoonright C$ clusters at $x \in X$, then the net f clusters at x.

(2) If $(D, \leq)$ does not have a well ordered cofinal subset then there exist a companion $(C, \preceq)$ of $(D, \leq)$ and a net $f : D \to X$ such that the companion sequence $f \upharpoonright C$ has a cluster point, but the net f does not have a cluster point.

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