DENSE METRIZABILITY

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Outline

1. Dense metrizability
2. Complete dense metrizability
3. Character of generic points
4. From Eberlein to Corson
5. Some examples old and new
6. A hierarchy of compact spaces
7. A spectrum of dense metrizability
8. Dense metrizability in powers
9. Cellularity versus density in powers
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Characterizing dense metrizability

Theorem (T., 1999)
The following are equivalent for a compact Hausdorff space $K$:

1. $K$ contains a dense metrizable subspace.
2. $K$ has a dense set of $G_\delta$ points and the generic ultrafilter of the regular open algebra of $K$ is countably generated.

Corollary (T., 1999)
The following are equivalent for a compactum $K$ with a dense set of $G_\delta$-points:

1. $K$ has a dense metrizable subspace.
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Theorem (Shapirovskii, 1980)

*Every Corson compactum $K$ has a dense set of $G_\delta$-points.*
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Proof.
Assume $K \subseteq \Sigma(I)$ for some index-set $I$.
Let $P_I$ be the standard $\sigma$-closed poset that forces $|I| = \aleph_1$. 
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Let $\mathbb{P}_I$ be the standard $\sigma$-closed poset that forces $|I| = \aleph_1$.
**Note that $K$ remains compact in the forcing extension of $\mathbb{P}_I$.**
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Moreover, $\mathbb{P}_I$ forces that $|K| = \aleph_1$. 
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**Note that $K$ remains compact in the forcing extension of $\mathbb{P}_I$.**

Moreover, $\mathbb{P}_I$ forces that $|K| = \aleph_1$.
Therefore $K$ has a $G_\delta$-point, a statement that is absolute between the universe and the forcing extension of $\mathbb{P}_I$. 

$\square$
Theorem (Bourgain, 1978)

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**Proof.**
Assume $K$ is a subspace of the space of Baire-class-1 functions on some Polish space $X$. 

Let $P$ be the standard $\sigma$-closed poset that forces $|X| = \aleph_1$. Note that forcing with $P$ does not change $X$, the set of Baire-class-1 functions on $X$ and the fact $K$ is sequentially compact. By a result of Rosenthal sequentially compact sets of Baire-class-1 functions are compact. Therefore in the forcing extension, the set $K$ remains compact. Thus $K$ is compact in the forcing extension and has cardinality at most $\aleph_1$. Thus, $K$ has a $G_\delta$-point in the forcing extension by $P$, a statement that is absolute between the universe and the forcing extension.
Theorem (Bourgain, 1978)

Every compact subset $K$ of the first Baire class has a dense set of $G_\delta$-points.

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By a result of Rosenthal sequentially compact sets of Baire-class-1 functions are compact. Therefore in the forcing extension, the set $K$ remains compact.
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Thus $K$ is compact in the forcing extension and has cardinality at most $\aleph_1$. Thus, $K$ has a $G_\delta$-point in the forcing extension by $\mathbb{P}$, a statement that is absolute between the universe and the forcing extension.
The first application of the method

Theorem (T., 1999)

If $K$ is a compact set of Baire-class-1 functions then the generic filter of the regular-open algebra of $K$ is countably generated.

Proof. (Sketch) Assume $K \subseteq \mathcal{B}_1(X)$ for some Polish space $X$.

Let $P_K = \text{RO}(K)$ and go to the forcing extension of $P_K$.

Let $\hat{X}$ be the metric completion of $X$.

Then every $f \in K$ naturally extends to $\hat{f} \in \mathcal{B}_1(\hat{X})$.

Then $\hat{K} = \{\hat{f} : f \in K\}$ is relatively compact in $\mathcal{B}_1(\hat{X})$.

Then the closure $\hat{K}$ is included in $\mathcal{B}_1(\hat{X})$.

The generic filter is generated by sets that form a free sequence of regular pairs of $\hat{K}$ and so it is countably generated.

Corollary (T., 1999)

Every compact set of Baire-class-1 functions has a dense metrizable subspace.
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**Theorem (T., 1999)**

*If \( K \) is a compact set of Baire-class-1 functions then the generic filter of the regular-open algebra of \( K \) is countably generated.*

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Let \( P_K = \mathcal{RO}(K) + \) and go to the forcing extension of \( P_K \).

Let \( \hat{\mathcal{X}} \) be the metric completion of \( \mathcal{X} \).

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Theorem (T., 1999)

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Then every $f \in K$ *naturally extends to* $\hat{f} \in B_1(\hat{X})$.

*Then* $\hat{K} = \{ \hat{f} : f \in K \}$ *is relatively compact in* $B_1(\hat{X})$.  

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**Corollary (T., 1999)**

*Every compact set of Baire-class-1 functions has a dense metrizable subspace.*
Compact spaces of functional analysis

K is a Eberlein compact if it is homeomorphic to a weakly compact subset of a Banach space.

K is a Talagrand compact if the Banach space C(K) with its weak topology is K-analytic (continuous image of a closed subset of the product of irrationals and a compact space).

K is a Gul'ko compact if the Banach space C(K) with its weak topology is countably determined (continuous image of a closed subset of the product of a set of irrationals and a compact space).

K is a Corson compact if it can be embedded in a Σ-Product of the real line.
$K$ is a **Eberlein compact** if it is homeomorphic to a weakly compact subset of a Banach space.
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$K$ is a **Corson compact** if it can be embedded in a $\Sigma$-Product of the real line.
An old example of a Corson compact space


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Theorem (T., 1978)

There is a first countable Corson compact space without dense metrizable subspace.
An old example of a Corson compact space

Theorem (T., 1978)

There is a first countable Corson compact space without dense metrizable subspace.

Proof.
(Sketch) Choose an everywhere branching Baire subtree of $\bigcup_{\alpha<\omega_1} \omega^\alpha$ with no uncountable branches and let

$$K_T = \{1_A : A \text{ is a path of } T\} \subseteq \{0, 1\}^T.$$
Sokolov’s characterization of Gul’ko compacta

A compactum $K$ is Gulko if it can be embedded into a Tychonov cube $\mathbb{R}$ in such a way that for some countable decomposition $I = \bigcup_{n<\omega} I_n$ of the index set $I$, we have that for every $x \in K$, if we let $N_x = \{n<\omega: |\text{supp}(x) \cap I_n| < \aleph_0\}$, then $I = \bigcup_{n \in N_x} I_n$. 
Sokolov’s characterization of Gul’ko compacta

Theorem (Sokolov, 1984)

A compactum $K$ is Gulko if it can be embedded into a Tychonov cube $\mathbb{R}^l$ in such a way that for some countable decomposition

$$l = \bigcup_{n<\omega} l_n$$

of the index set $I$, we have that for every $x \in K$, if we let

$$N_x = \{n < \omega : |\text{supp}(x) \cap l_n| < \aleph_0\},$$

then

$$l = \bigcup_{n \in N_x} l_n.$$
Theorem (Sokolov, 1984)

A compactum \( K \) is Gul’ko if it has a weakly \( \sigma \)-point-finite \( T_0 \)-separating cover by co-zero sets, i.e. a \( T_0 \)-separating cover \( \mathcal{U} \) by co-zero sets which has a decomposition

\[
\mathcal{U} = \bigcup_{n<\omega} \mathcal{U}_n
\]

such that for every \( x \in K \), if we let

\[
N_x = \{ n < \omega : \text{ord}(x, \mathcal{U}_n) < \aleph_0 \},
\]

then \( \mathcal{U} = \bigcup_{n \in N_x} \mathcal{U}_n \).
Two classical results

Theorem (Namioka, 1974)
Every Eberlein compactum has a dense completely metrizable subspace.
Proof.
(Hint). Use Namioka's joint versus separate continuity theorem

Theorem (Leiderman, 1985; Gruenhage, 1987)
Every Gul'ko compactum has a dense completely metrizable subspace.
Proof.
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Proof.
(Hint). Use Sokolov’s characterization theorem.
Definition

For a cardinal $\theta$, we say that a compact subset $K$ of the Tychonov cube $\mathbb{R}^I$ has the property $E_2(\theta)$ if there is a sequence $I_n (n < \omega)$ of subsets of $I$ such that if for $x \in K$, we let $N_x = \{n < \omega : |\text{supp}(x) \cap I_n| < \aleph_0\}$, then $|I \setminus \bigcup_{n \in N_x} I_n| < \theta$.

Remark (1) $E_2(1)$ is the class of Gul'ko compacta.

Remark (2) $E_2(\beth_1)$ is included in the class of Corson compacta.

Remark (3) $E_2(\beth_1)$ was first considered by Leiderman (2012) under the name almost Gul'ko compact spaces.
A hierarchy of compact spaces

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**Definition**
For a cardinal $\theta$, we say that a compact subset $K$ of the Tychonov cube $\mathbb{R}^I$ has the property $\mathcal{E}_2(\theta)$ if there is a sequence $I_n (n < \omega)$ of subsets of $I$ such that if for $x \in K$, we let

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**Remark**
(1) $\mathcal{E}_2(1)$ is the class of Gul’ko compacta.
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For a cardinal $\theta$, we say that a compact subset $K$ of the Tychonov cube $\mathbb{R}^I$ has the property $E_2(\theta)$ if there is a sequence $I_n$ ($n < \omega$) of subsets of $I$ such that if for $x \in K$, we let

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(1) $E_2(1)$ is the class of Gul’ko compacta.
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Two examples in $\mathcal{E}_2(2) \setminus \mathcal{E}_2(1)$

Example (Leiderman, 1985)

Let $I = [0, 1]$ and let $K_L = \{ A : A \subseteq I$ and $(\exists b \in I) \sum_{a \in A} |b - a| \leq 1 \}$. Then $K_L \in \mathcal{E}_2(2)$ by letting $I_n (n < \omega)$ be an enumeration of all intervals of $I$ with rational end-points.

Example (Argyros-Marcourakis, 1993)

Call a subset $A$ of $I = [0, 1]$ admissible if for every finite subset $a_1 < \cdots < a_n$ of $A$, we have that $a_n - a_m < \frac{1}{m}$ for all $m < n$.

Let $K_{AM} = \{ A : A \text{ admissible subset of } I \}$. Then $K_{AM} \in \mathcal{E}_2(2)$ by letting again $I_n (n < \omega)$ be an enumeration of all intervals of $I$ with rational end-points.
Two examples in $\mathcal{E}_2(2) \setminus \mathcal{E}_2(1)$

Example (Leiderman, 1985)
Let $I = [0, 1]$ and let

$$K_L = \{1_A : A \subseteq I \text{ and } (\exists b \in I) \sum_{a \in A} |b - a| \leq 1\}.$$

Then $K_L \in \mathcal{E}_2(2)$ by letting $I_n (n < \omega)$ be an enumeration of all intervals of $I$ with rational end-points.

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Then $K_{AM} \in \mathcal{E}_2(2)$ by letting again $I_n (n < \omega)$ be an enumeration of all intervals of $I$ with rational end-points.
A Corson compactum in $\mathcal{E}_2(c^+) \setminus \mathcal{E}_2(c)$
A Corson compactum in $\mathcal{E}_2(c^+) \setminus \mathcal{E}_2(c)$

Example

Let $T$ to be the tree of all closed subsets of a stationary subset $E$ of $\omega_1$ whose complement $\omega_1 \setminus E$ is also stationary. The Corson compactum

$$K_T = \{1_A : A \text{ is a path of } T\}$$

has no metrizable subspaces and $K_T \not\in \mathcal{E}_2(c)$. 
A Corson compactum in $\mathcal{E}_2(c^+) \setminus \mathcal{E}_2(c)$

Example

Let $T$ to be the tree of all closed subsets of a stationary subset $E$ of $\omega_1$ whose complement $\omega_1 \setminus E$ is also stationary. The Corson compactum

$$K_T = \{1_A : A \text{ is a path of } T\}$$

has no metrizable subspaces and $K_T \not\in \mathcal{E}_2(c)$.

Question

For which $\theta$ do we have that every compactum in $\mathcal{E}_2(\theta)$ has a metrizable subspace?
Theorem (T., 2022)

There is a compact subset $K$ of $\Sigma^b(I)$ for some index set $I$ of cardinality $b$ such that $K \in E^2(b)$ and $K$ has no dense metrizable subspace.

Corollary (T., 2022)

If $b = \aleph_1$ there is a (Corson) compactum in $E^2(\aleph_1)$ without a dense metrizable subspace.
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Fix a set $I \subseteq \omega^\omega$ of increasing functions well-ordered by $\prec^*$ in order type $b$ and unbounded in $(\omega^\omega, \prec^*)$. 
Fix a set $I \subseteq \omega^\omega$ of increasing functions well-ordered by $<^*$ in order type $b$ and unbounded in $(\omega^\omega, <^*)$.

For $a \neq b$ in $I$, let

$$D(a, b) = \{n < \omega : a(n) \neq b(n)\}.$$
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For $m, n \in D(a, b)$, set

$mE(a, b)n$ if either $a >_{[m,n]} b$ or $b >_{[m,n]} a$. 

Finally, let

$$\osc^*(a, b) = |D(a, b)/E(a, b)|.$$ 

and

$$\osc(a, b) = \osc(a \upharpoonright k, b \upharpoonright k),$$ 

where $k$ is the minimum of the first relatively large equivalence class in $D(a, b)/E(a, b)$. 

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A crucial property of the oscillation mapping

For every positive integers $k$ and $\ell$ and every family $F$ of pairwise disjoint subsets of $I$ of size $\ell$ there exist $p \neq q$ in $F$ such that $\text{osc}^* (p(i), q(i)) + 1 = \text{osc} (p(i), q(i)) = k$ for all $i < \ell$.

Define $c : [I]^2 \to \{0, 1\}$ by letting $c(\{a, b\}) = 0$ if and only if $\text{osc} (a, b)$ is even.

Let $K = \{A : A \subseteq I \text{ and } c(A) = \{0\}\}$. 
A crucial property of the oscillation mapping

(o) For every positive integers $k$ and $\ell$ and every family $\mathcal{F}$ of pairwise disjoint subsets of $I$ of size $\ell$ there exist $p \neq q$ in $\mathcal{F}$ such that

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$$K = \{1_A : A \subseteq I \text{ and } c[[A]^2] = \{0\}\}.$$
Properties of $K$

(1) For $K$ but $K$ has no cellular family of open subsets of cardinality $b$. Thus, $K$ has no dense metrizable subspace.

(2) Let $s_n (n < \omega)$ be an enumeration of $\omega < \omega$. For $n < \omega$, set $I_n = \{a \in I: s_n \sqsubseteq a\}$. Then ($I_n: n < \omega$) establishes the fact that $K \in E_2 (b)$.

Namely, if for $x_1 = 1 A$ in $K$, we let $N_x = \{n < \omega: |A \cap I_n| < \aleph_0\}$, then $I \setminus \bigcup_{n \in N_x} I_n$ has cardinality $< b$. 
Properties of $K$

(1) $d(K) = \mathfrak{b}$ but $K$ has no cellular family of open subsets of cardinality $\mathfrak{b}$. Thus, $K$ has no dense metrizable subspace.

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Properties of $K$

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then $I \setminus \bigcup_{n \in N_x} I_n$ has cardinality $< b$. 
The main result

Theorem (T., 2022) The generic ultrafilter of every compactum in $E^{\aleph_0}$ is countably generated.

Corollary (T., 2022) Every compactum in the class $E^{\aleph_0}$ contains a dense metrizable subspace.
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Theorem (T., 2022)

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The main result

**Theorem (T., 2022)**

The generic ultrafilter of every compactum in $E_2(\aleph_0)$ is countably generated.

**Corollary (T., 2022)**

Every compactum in the class $E_2(\aleph_0)$ contains a dense metrizable subspace.
Sketch of a proof

Fix a compact subset $K$ of some $\Sigma$-product $\Sigma(I)$ and assume that the generic ultra-filter of the regular-open algebra $\text{RO}(K)$ is not countably generated and go towards showing $K \not\in E_2(\mathbb{N})$.

We assume that $I$ well-ordered and replacing $I$ by an initial segment $\Gamma$ and $K$ by its projection to $\Sigma(\Gamma)$, we may assume that every element of $\text{RO}(K)^+$ forces that $I$ has uncountable cofinality.

Let $\dot{x}_G$ be the $\text{RO}(K)^+$-name for the generic point of $K$, the intersection of closures of elements of the generic filter $\dot{G}$ and let $\dot{J}$ be the $\text{RO}(K)^+$-name for the set $\{\gamma \in I : (\exists n) \{y \in K : |y(\gamma)| > 1/n\} \in \dot{G}\}$.

Note that our assumption in particular means that every member of $\text{RO}(K)^+$ forces that $\dot{J}$ is a cofinal subset of $I$. 
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Note that our assumption in particular means that every member of $\text{RO}(K)^+$ forces that $\dot{J}$ is a cofinal subset of $I$. 
Fix a sequence \( l_n (n < \omega) \) of subsets of \( I \).
We shall find \( x \in K \) such that \( I \setminus \bigcup_{n \in N_x} l_n \) is infinite, where
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Let $\dot{N}$ be the $\text{RO}(K)^+$-name for the set of all $n < \omega$ such that
$l_n \cap \dot{J}$ is bounded in $I$. 
Fix a sequence \( I_n (n < \omega) \) of subsets of \( I \). We shall find \( x \in K \) such that \( I \setminus \bigcup_{n \in N_x} I_n \) is infinite, where \( N_x = \{ n < \omega : \text{supp}(x) \cap I_n \text{ is finite} \} \).

Let \( \dot{N} \) be the \( \text{RO}(K)^+ \)-name for the set of all \( n < \omega \) such that \( I_n \cap J \) is bounded in \( I \).

Let \( P \) be the collection of all finite partial mappings \( p \) from \( I \) to open intervals of \( \mathbb{R} \) with end points in \( \mathbb{Q} \) such that for every \( i \in \text{dom}(p) \), the interval \( p(i) \) is either centered at 0 and both of its end points are strictly above or strictly below 0 and such that

\[
O(p) = \{ x \in K : \forall i \in \text{dom}(p) \ x(i) \in p(i) \}
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Let \( \mathcal{P} \) be the collection of all finite partial mappings \( p \) from \( I \) to open intervals of \( \mathbb{R} \) with end points in \( \mathbb{Q} \) such that for every \( i \in \text{dom}(p) \), the interval \( p(i) \) is either centered at 0 and both of its end points are strictly above or strictly below 0 and such that

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O(p) = \{ x \in K : \forall i \in \text{dom}(p) \ x(i) \in p(i) \}
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is a nonempty open subset of \( K \). Note that \( O(p) \ (p \in \mathcal{P}) \) is a dense subset of RO\((K)^+\). For \( p \in \mathcal{P} \), let

\[
\text{supp}(p) = \{ i \in \text{dom}(p) : 0 \not\in p(i) \}.
\]
Fix $p_0 \in \mathbb{P}$ and $\alpha_0 \in \Gamma$ such that $O(p_0)$ forces that $\alpha_0$ is an upper bound of the set $\bigcup_{n \in \hat{\mathbb{N}}} I_n \cap \hat{J}$. We may assume that $\alpha_0 \in \text{dom}(p_0)$. 
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Fix an enumeration $n_k$ ($k < \omega$) of $\omega$ such that every $n < \omega$ is equal to $n_k$ for infinitely many $k$. 
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Starting from $p_0$ and $\alpha_0$, recursively on $k < \omega$, we define an increasing sequence $p_k$ of elements of $\mathbb{P}$ and an increasing sequence $\alpha_k$ of ordinals from $I$ such that for all $k$:
Fix \( p_0 \in \mathbb{P} \) and \( \alpha_0 \in \Gamma \) such that \( O(p_0) \) forces that \( \alpha_0 \) is an upper bound of the set \( \bigcup_{n \in \dot{N}} I_n \cap J \). We may assume that \( \alpha_0 \in \text{dom}(p_0) \).

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If there is \( q \in \mathbb{P} \) extending \( p_k \) such that \( O(q) \) forces that \( n_k \in \dot{N} \), we choose \( p_{k+1} \) to extend such \( q \) and have an \( \alpha_{k+1} > \alpha_k \) in \( \text{supp}(p_{n+1}) \).
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If such a \( q \) cannot be found, we have that \( O(p_k) \) forces \( n_k \not\in \dot{N} \), so we can then find \( \alpha_{k+1} > \alpha_k \) in \( I_{n_k} \) and \( p_{k+1} \) extending \( p_k \) such that \( \alpha_{k+1} \in \text{supp}(p_{k+1}) \).
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Pick an element $x$ in the intersection $\bigcap_{k<\omega} \overline{O(p_k)}$. Let $A = \{\alpha_k : k < \omega\}$. Note that $A \subseteq \text{supp}(x)$. 
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**We claim that** $I_n \cap A = \emptyset$ **for all** $n \in N_x$. 

It follows that $K \not\in E_2(\aleph_0)$. The proof of the main result is finished.
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This shows that $A$ is then an infinite subset of $I \setminus \bigcup_{n \in N_x} I_n$ and therefore that $|I \setminus \bigcup_{n \in N_x} I_n| \geq \aleph_0$, as required.
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It follows that $K \notin \mathcal{E}_2(\aleph_0)$.

The proof of the main result is finished.
Theorem (Leiderman-Spadaro-T., 2021)

The following are equivalent for every Corson compact space $K$:

1. $K^\omega$ has a dense metrizable subspace.
2. $K^\omega$ has a cellular family of open sets of cardinality $d(K)$.

Proof. (Sketch) To prove the implication from (2) to (1), it suffices to prove that the generic ultrafilter of the forcing notion $O(K^\omega)^+$ is countably generated. For this, we show that $O(K^\omega)^+$ forces that $K$ and therefore $K^\omega$ has countable $\pi$-basis.

Fix a $\pi$-basis $P$ of $K$ of cardinality $d(K)$.

Partition $\omega$ into countably many infinite sets $I_n$ ($n<\omega$). Our assumption allows us to fix for each $n<\omega$ a cellular family $C_n$ of cardinality $d(K)$ of finitely supported open sets with supports all included in the infinite set $I_n$. 
Dense metrizability in powers

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Fix a $\pi$-basis $\mathcal{P}$ of $K$ of cardinality $d(K)$.

Partition $\omega$ into countably many infinite sets $I_n (n < \omega)$.

Our assumption allows us to fix for each $n < \omega$ a cellular family $C_n$ of cardinality $d(K)$ of finitely supported open sets with supports all included in the infinite set $I_n$. 
For each $n < \omega$, we fix a bijection $f_n : \mathcal{P} \rightarrow C_n$. 

Let $\dot{G}$ be the $\mathcal{O}(K^\omega)^+-$name of the generic ultrafilter of $\mathcal{O}(K^\omega)^+$. Using this we describe a $\mathcal{O}(K^\omega)^+-$name $\dot{g} : \mathcal{P} \rightarrow \omega$ for an injection, as follows. Given $U \in \mathcal{P}$, we let $\dot{g}(U)$ to be the minimal $n < \omega$ such that $f_n(U) \in \dot{G}$. Note that since $C_n$ is a cellular family and since $f_n$ is an injection, no two different $U$ and $V$ in $\mathcal{P}$ get mapped to the same $n$, so indeed $\dot{g}$ is an injection. To see that $\dot{g}$ is indeed a name for a function with domain $\mathcal{P}$, fix a member $V$ of $\mathcal{O}(K^\omega)^+$ and $U \in \mathcal{P}$. By going to a subset, we may assume, $V$ has finite support. Pick $n < \omega$ so that $I_n$ does not intersect the support of $V$. Then $V$ and $f_n(U)$ are compatible, so their intersection $V \cap f_n(U)$ is a refinement of $V$ forcing that $\dot{g}(U)$ is defined. Since $V$ was arbitrary, this finishes the proof.
For each $n < \omega$, we fix a bijection $f_n : \mathcal{P} \rightarrow \mathcal{C}_n$.
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For each \( n < \omega \), we fix a bijection \( f_n : \mathcal{P} \rightarrow \mathcal{C}_n \).
Let \( \mathcal{G} \) be the \( O(K^\omega)^+ \)-name of the generic ultrafilter of \( O(K^\omega)^+ \).

Using this we describe a \( O(K^\omega)^+ \)-name \( \dot{g} : \mathcal{P} \rightarrow \omega \) for an injection, as follows.

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Question
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Theorem (Leiderman-Spadaro-T., 2021)

*If there is a locally countable family of countable sets of cardinality bigger than the cardinality of its union, then there is a Corson compactum $K$ such that $K^\omega$ has no dense metrizable subspace.*
Sketch of the construction
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The assumption allows us to find a cardinal $\kappa$ and a subset $I$ of $\kappa^\omega$ of cardinality bigger than $\kappa$ such that

$$T(A) = \{ a \upharpoonright n : a \in A, n < \omega \}$$

is uncountable for every uncountable $A \subseteq I$. 
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Call a subset $A$ of $I$ **binary** if the tree $T(A)$ is binary, i.e., every node of $T(A)$ has at most two immediate successors.
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The assumption allows us to find a cardinal \( \kappa \) and a subset \( I \) of \( \kappa^\omega \) of cardinality bigger than \( \kappa \) such that

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**By our choice of \( I \), no uncountable subset of \( I \) is binary.**
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The assumption allows us to find a cardinal $\kappa$ and a subset $I$ of $\kappa^\omega$ of cardinality bigger than $\kappa$ such that

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Call a subset $A$ of $I$ binary if the tree $T(A)$ is binary, i.e., every node of $T(A)$ has at most two immediate successors.

By our choice of $I$, no uncountable subset of $I$ is binary.

Let

$$K = \{ 1_A : A \subseteq I \text{ is binary} \}.$$
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The assumption allows us to find a cardinal $\kappa$ and a subset $I$ of $\kappa^\omega$ of cardinality bigger than $\kappa$ such that

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Let

$$K = \{ 1_A : A \subseteq I \text{ is binary} \}.$$

Then $K$ is Corson, $d(K^\omega) = \kappa^+$ and $c(K^\omega) = \kappa$. 
Sketch of the construction

The assumption allows us to find a cardinal $\kappa$ and a subset $I$ of $\kappa^\omega$ of cardinality bigger than $\kappa$ such that

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is uncountable for every uncountable $A \subseteq I$.

Call a subset $A$ of $I$ **binary** if the tree $T(A)$ is binary, i.e., every node of $T(A)$ has at most two immediate successors.

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Let

$$K = \{ 1_A : A \subseteq I \text{ is binary} \}.$$  

Then $K$ is Corson, $d(K^\omega) = \kappa^+$ and $c(K^\omega) = \kappa$. So $K^\omega$ has no dense metrizable subspace.
Theorem (T., 2022)

There exist two compact subsets \( K_0 \) and \( K_1 \) of \( \Sigma^b(I) \), both belonging to the class \( E_2^b \) such that neither of the infinite powers \( K_0^\omega \) and \( K_1^\omega \) has a dense metrizable subspace but their product does have a dense metrizable subspace.

Proof. (Sketch) As before we fix a subset \( I \) of \( \omega^\omega \) consisting of increasing mappings from \( \omega \) into \( \omega \) such that \( I \) is well-ordered by \( <^* \) in order type \( b \) and such that \( I \) is unbounded in \( (\omega^\omega, <^*) \). Consider the oscillation mappings \( \text{osc} : [I]^{2} \to \omega \) and \( \text{osc}^* : [I]^{2} \to \omega \) on \( I \) and the projection \( c : [I]^{2} \to 2 \).

Let \( K_0 = \{ 1^A : A \subseteq I, c[A] = \{ 0 \} \} \) and \( K_1 = \{ 1^A : A \subseteq I, c[A] = \{ 1 \} \} \).
Theorem (T., 2022)

There exist two compact subspaces $K_0$ and $K_1$ of $\Sigma_b(I)$, both belonging to the class $\mathcal{E}_2(b)$ such that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a dense metrizable subspace but their product does have a dense metrizable subspace.
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Proof.
(Sketch) As before we fix a subset $I$ of $\omega^\omega$ consisting of increasing mappings from $\omega$ into $\omega$ such that $I$ is well-ordered by $<^*$ in order type $b$ and such that $I$ is unbounded in $(\omega^\omega, <^*)$. and consider the oscillation mappings $\text{osc} : [I]^2 \to \omega$ and $\text{osc}^* : [I]^2 \to \omega$ on $I$ and the projection $c : [I]^2 \to 2$. 

\textit{b-example}
Theorem (T., 2022)

There exist two compact subspaces $K_0$ and $K_1$ of $\Sigma_b(I)$, both belonging to the class $\mathcal{E}_2(b)$ such that neither of the infinite powers $K_0^{\omega}$ and $K_1^{\omega}$ has a dense metrizable subspace but their product does have a dense metrizable subspace.

Proof.

(Sketch) As before we fix a subset $I$ of $\omega^\omega$ consisting of increasing mappings from $\omega$ into $\omega$ such that $I$ is well-ordered by $<^*$ in order type $b$ and such that $I$ is unbounded in $(\omega^\omega, <^*)$. and consider the oscillation mappings $\text{osc} : [I]^2 \to \omega$ and $\text{osc}^* : [I]^2 \to \omega$ on $I$ and the projection $c : [I]^2 \to 2$. Let

\[ K_0 = \{1_A : A \subseteq I, c[[A]^2] = \{0\}\} \text{ and } K_1 = \{1_A : A \subseteq I, c[[A]^2] = \{1\}\}. \]
Then as before $K_0$ and $K_1$ belong to $\mathcal{E}_2(b)$. 
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The crucial property (o) of the oscillation mapping shows that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a cellular family of open sets of cardinality $\mathfrak{b}$.
Then as before $K_0$ and $K_1$ belong to $E_2(b)$.

The crucial property (o) of the oscillation mapping shows that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a cellular family of open sets of cardinality $b$.

It follows that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a dense metrizable subspace.
Then as before $K_0$ and $K_1$ belong to $\mathcal{E}_2(\mathfrak{b})$.

The crucial property (o) of the oscillation mapping shows that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a cellular family of open sets of cardinality $\mathfrak{b}$.

It follows that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a dense metrizable subspace.

It remains to prove that the product $K_0^\omega \times K_1^\omega$ does have a dense metrizable subspace.
Then as before $K_0$ and $K_1$ belong to $\mathcal{E}_2(\mathfrak{b})$.

The crucial property (o) of the oscillation mapping shows that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a cellular family of open sets of cardinality $\mathfrak{b}$.

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It remains to prove that the product $K_0^\omega \times K_1^\omega$ does have a dense metrizable subspace.

Since $K_0^\omega \times K_1^\omega = (K_0 \times K_1)^\omega$ it suffices to show that the product $K_0 \times K_1$ has a cellular family of open sets of cardinality $\mathfrak{b} = d(K_0 \times K_1)$. 
For $a \in I$ and $i < 2$, set

$$[a]_i = \{1_A \in K_i : a \in A\}.$$
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$$[a]_i = \{1_A \in K_i : a \in A\}.$$

Then for all $a \in I$ and $i < 2$, the $[a]_i$ is a nonempty basic open set of $K_i$ and the family

$$\mathcal{F} = \{[a]_0 \times [a]_1 : a \in I\}$$

is a cellular family of cardinality $b$ of nonempty basic open subsets of the product $K_0 \times K_1$. 

Corollary (T., 2022) If $b = \aleph_1$ there exist two compacta $K_0$ and $K_1$ in $E_2(\aleph_1)$ such that neither of the infinite powers $K_\omega_0$ and $K_\omega_1$ has a dense metrizable subspace but their product does have a dense metrizable subspace.
For $a \in I$ and $i < 2$, set

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Then for all $a \in I$ and $i < 2$, the $[a]_i$ is a nonempty basic open set of $K_i$ and the family

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**Corollary (T., 2022)**

If $b = \aleph_1$ there exist two compacta $K_0$ and $K_1$ in $\mathcal{E}_2(\aleph_1)$ such that neither of the infinite powers $K_0^\omega$ and $K_1^\omega$ has a dense metrizable subspace but their product does have a dense metrizable subspace.
References


