More on the structure theory of compact subsets of the first Baire class

Stevo Todorcevic

Prague, July 28, 2016

Outline

Outline

- Compact sets of the first Baire class

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta
- G_δ-points

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta
- G_δ-points
- Dense metrizable subspaces

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta
- G_δ-points
- Dense metrizable subspaces
- The slit interval, the duplicate, and the Cantor space

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta
- G_δ-points
- Dense metrizable subspaces
- The slit interval, the duplicate, and the Cantor space
- Separable compacta of the first Baire class

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta
- G_δ-points
- Dense metrizable subspaces
- The slit interval, the duplicate, and the Cantor space
- Separable compacta of the first Baire class
- Basis problem for separable compacta

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta
- G_δ-points
- Dense metrizable subspaces
- The slit interval, the duplicate, and the Cantor space
- Separable compacta of the first Baire class
- Basis problem for separable compacta
- Open degrees and cozero degrees

Outline

- Compact sets of the first Baire class
- Baire-class-1 and dual balls of Banach spaces
- Topological properties of BC1-compacta
- G_δ-points
- Dense metrizable subspaces
- The slit interval, the duplicate, and the Cantor space
- Separable compacta of the first Baire class
- Basis problem for separable compacta
- Open degrees and cozero degrees
- Applications

Compact sets of the first Baire class

Given a Polish space X, we let $\mathcal{B}_1(X)$ denote the collection of Baire-class-1 functions on X with the topology induced from $\mathbb{R}^X$.

Compact sets of the first Baire class

Given a Polish space X, we let $\mathcal{B}_1(X)$ denote the collection of Baire-class-1 functions on X with the topology induced from $\mathbb{R}^X$.
We are interested in **compact subspaces** of $\mathcal{B}_1(X)$.
Let **BC1** denote this class of compacta.

Compact sets of the first Baire class

Given a Polish space X, we let $\mathcal{B}_1(X)$ denote the collection of Baire-class-1 functions on X with the topology induced from $\mathbb{R}^X$.
We are interested in **compact subspaces** of $\mathcal{B}_1(X)$.
Let **BC1** denote this class of compacta.

Theorem (Rosethal 1977, Odell-Rosenthal 1979)

Let E be a separable Banach space.

- $\ell_1 \not\hookrightarrow E$ *iff* $B_{E^{**}} \subseteq \mathcal{B}_1(B_{E^*})$.
- $\ell_1 \not\hookrightarrow E$ *iff* B_E *is sequentially dense in* $B_{E^{**}}$.

Compact sets of the first Baire class

Given a Polish space X, we let $\mathcal{B}_1(X)$ denote the collection of Baire-class-1 functions on X with the topology induced from $\mathbb{R}^X$.
We are interested in **compact subspaces** of $\mathcal{B}_1(X)$.
Let **BC1** denote this class of compacta.

Theorem (Rosethal 1977, Odell-Rosenthal 1979)

Let E be a separable Banach space.

- $\ell_1 \not\hookrightarrow E$ *iff* $B_{E^{**}} \subseteq \mathcal{B}_1(B_{E^*})$.
- $\ell_1 \not\hookrightarrow E$ *iff* B_E *is sequentially dense in* $B_{E^{**}}$.

Example

- *If E is a separable Banach space such that $\ell_1 \not\hookrightarrow E$ then $B_{E^{**}}$ is a separable compact set of the first Baire class.*

Compact sets of the first Baire class

Given a Polish space X, we let $\mathcal{B}_1(X)$ denote the collection of Baire-class-1 functions on X with the topology induced from $\mathbb{R}^X$.
We are interested in **compact subspaces** of $\mathcal{B}_1(X)$.
Let **BC1** denote this class of compacta.

Theorem (Rosethal 1977, Odell-Rosenthal 1979)

Let E be a separable Banach space.

- $\ell_1 \not\hookrightarrow E$ *iff* $B_{E^{**}} \subseteq \mathcal{B}_1(B_{E^*})$.
- $\ell_1 \not\hookrightarrow E$ *iff* B_E *is sequentially dense in* $B_{E^{**}}$.

Example

- *If E is a separable Banach space such that $\ell_1 \not\hookrightarrow E$ then $B_{E^{**}}$ is a separable compact set of the first Baire class.*
- *The* **Helly space** *of monotone maps from $[0,1]$ into $[0,1]$ is another separable compact convex set of the first Baire class. Note that Helly space is moreover* **first countable**.

Topological properties

Theorem (Rosenthal, 1977)

Every compact set of the first Baire class is **countably tight** *and* **sequentially compact**.

Topological properties

Theorem (Rosenthal, 1977)

Every compact set of the first Baire class is **countably tight** *and* **sequentially compact**.

Theorem (Bourgain-Fremlin-Talagrand 1978)

Every compact set of the first Baire class has the **Fréchet-Urysohn property**

Topological properties

Theorem (Rosenthal, 1977)

Every compact set of the first Baire class is **countably tight** *and* **sequentially compact**.

Theorem (Bourgain-Fremlin-Talagrand 1978)

Every compact set of the first Baire class has the **Fréchet-Urysohn property**

Theorem (Bourgain 1978)

Every compact set K of the first Baire class has a G_δ-point

Topological properties

Theorem (Rosenthal, 1977)

Every compact set of the first Baire class is **countably tight** *and* **sequentially compact**.

Theorem (Bourgain-Fremlin-Talagrand 1978)

Every compact set of the first Baire class has the **Fréchet-Urysohn property**

Theorem (Bourgain 1978)

Every compact set K of the first Baire class has a G_δ-point

Question (Bourgain 1978)

Is the set of G_δ-points comeager in K?

Topological properties

Theorem (Rosenthal, 1977)

Every compact set of the first Baire class is **countably tight** *and* **sequentially compact**.

Theorem (Bourgain-Fremlin-Talagrand 1978)

Every compact set of the first Baire class has the **Fréchet-Urysohn property**

Theorem (Bourgain 1978)

Every compact set K of the first Baire class has a G_δ-point

Question (Bourgain 1978)

Is the set of G_δ-points comeager in K?

Theorem (T., 1999)

Every compact set of the first Baire class has a dense metrizable subspace.

Basis problem for BC1-compacta

Question

Is there a **finite list** $\mathcal{C}$ *of BC1-compacta that* **determines** *the class of metrizable compacta?*

Basis problem for BC1-compacta

Question

Is there a **finite list** $\mathcal{C}$ *of BC1-compacta that* **determines** *the class of metrizable compacta?*

Here **determines** refers to the specific condition:

A BC1-compactum is metrizable iff it contains no copy of any space from $\mathcal{C}$.

Basis problem for BC1-compacta

Question

Is there a **finite list** $\mathcal{C}$ *of BC1-compacta that* **determines** *the class of metrizable compacta?*

Here **determines** refers to the specific condition:
A BC1-compactum is metrizable iff it contains no copy of any space from $\mathcal{C}$.

Example

- *The* **split interval** $S(I)$ *is a BC1-compactum which contains no uncountable metrizable subspace.*

Basis problem for BC1-compacta

Question

Is there a **finite list** $\mathcal{C}$ *of BC1-compacta that* **determines** *the class of metrizable compacta?*

Here **determines** refers to the specific condition:
A BC1-compactum is metrizable iff it contains no copy of any space from $\mathcal{C}$.

Example

- *The* **split interval** $S(I)$ *is a BC1-compactum which contains no uncountable metrizable subspace.*
- *The* **Alexandrov duplicate** $D(I)$ *of the unit interval* I *is an example of a* **first countable** *non-metrizable BC1-compactum.*

Basis problem for BC1-compacta

Question

Is there a **finite list** $\mathcal{C}$ *of BC1-compacta that* **determines** *the class of metrizable compacta?*

Here **determines** refers to the specific condition:
A BC1-compactum is metrizable iff it contains no copy of any space from $\mathcal{C}$.

Example

- *The* **split interval** $S(I)$ *is a BC1-compactum which contains no uncountable metrizable subspace.*
- *The* **Alexandrov duplicate** $D(I)$ *of the unit interval* I *is an example of a* **first countable** *non-metrizable BC1-compactum.*
- *The* **Cantor tree compactum** $C(2^{<\mathbb{N}}) = 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\}$ *is a non-metrizable BC1-compactum with the point at infinity as the single non* G_δ*-point*

G_δ points and the Cantor tree compactum

G_δ points and the Cantor tree compactum

Theorem (T., 1999)

Suppose that K is a BC1-compactum, D a dense subset of K, and that x is a non-G_δ-point of K. Then there is a topological embedding

$$\Phi : 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\} \to K$$

of the Cantor tree space $C(2^{<\mathbb{N}})$ into K such that

$$\Phi[2^{<\mathbb{N}}] \subseteq D \text{ and } \Phi(\infty) = x.$$

G_δ points and the Cantor tree compactum

Theorem (T., 1999)

Suppose that K is a BC1-compactum, D a dense subset of K, and that x is a non-G_δ-point of K. Then there is a topological embedding

$$\Phi : 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\} \to K$$

of the Cantor tree space $C(2^{<\mathbb{N}})$ into K such that

$$\Phi[2^{<\mathbb{N}}] \subseteq D \text{ and } \Phi(\infty) = x.$$

Theorem

A BC1-compactum is **first countable** *iff it contains no $C(2^{<\mathbb{N}})$.*

G_δ points and the Cantor tree compactum

Theorem (T., 1999)

Suppose that K is a BC1-compactum, D a dense subset of K, and that x is a non-G_δ-point of K. Then there is a topological embedding

$$\Phi : 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\} \to K$$

of the Cantor tree space $C(2^{<\mathbb{N}})$ into K such that

$$\Phi[2^{<\mathbb{N}}] \subseteq D \text{ and } \Phi(\infty) = x.$$

Theorem

A BC1-compactum is **first countable** *iff it contains no $C(2^{<\mathbb{N}})$.*

Application:

G_δ points and the Cantor tree compactum

Theorem (T., 1999)

Suppose that K is a BC1-compactum, D a dense subset of K, and that x is a non-G_δ-point of K. Then there is a topological embedding

$$\Phi : 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\} \to K$$

of the Cantor tree space $C(2^{<\mathbb{N}})$ into K such that

$$\Phi[2^{<\mathbb{N}}] \subseteq D \text{ and } \Phi(\infty) = x.$$

Theorem

A BC1-compactum is **first countable** *iff it contains no $C(2^{<\mathbb{N}})$.*

Application:

Theorem (Argyros-Dodos-Kanellopoulos 2008)

Every dual Banach space has a **separable quotient.**

Separable BC1-compacta

Theorem (T., 1999)

Every separable **nonmetrizable** *BC1-compactum contains a topological copy of one of the three compacta*

$$S(2^{\mathbb{N}}), D_{\mathrm{sep}}(2^{\mathbb{N}}), C(2^{<\mathbb{N}}).$$

where $D_{\mathrm{sep}}(2^{\mathbb{N}})$ *is the natural separable version of the Alexandrov duplicate of the Cantor set.*

Separable BC1-compacta

Theorem (T., 1999)

Every separable **nonmetrizable** *BC1-compactum contains a topological copy of one of the three compacta*

$$S(2^{\mathbb{N}}), D_{\mathrm{sep}}(2^{\mathbb{N}}), C(2^{<\mathbb{N}}).$$

where $D_{\mathrm{sep}}(2^{\mathbb{N}})$ *is the natural separable version of the Alexandrov duplicate of the Cantor set.*

Question

Can one characterize some natural classes of separable BC1-compacta using some of the three basic compacta?

Separable BC1-compacta

Theorem (T., 1999)

Every separable **nonmetrizable** *BC1-compactum contains a topological copy of one of the three compacta*

$$S(2^{\mathbb{N}}), D_{\mathrm{sep}}(2^{\mathbb{N}}), C(2^{<\mathbb{N}}).$$

where $D_{\mathrm{sep}}(2^{\mathbb{N}})$ *is the natural separable version of the Alexandrov duplicate of the Cantor set.*

Question

Can one characterize some natural classes of separable BC1-compacta using some of the three basic compacta?

Theorem (T., 1999)

Every **hereditarily separable** *non-metrizable BC1-compactum contains* $S(2^{\mathbb{N}})$.

Open degrees

Definition

Fix a compactum K. The **open degree** *of K, if it exists, is the least positive integer n for which we can find a countable family $\mathcal{F}$ of open subsets of K such that for every one-to-one $(n+1)$-sequence $x_0, \ldots, x_n \in K$ there exist $V_0, \ldots, V_n \in \mathcal{F}$ such that:*

- $x_i \in V_i$ *for all* $i \leq n$,
- $\bigcap_0^n V_i = \emptyset$.

Put $\mathrm{odeg}(K) = \infty$ *if such n does not exist.*

Open degrees

Definition

Fix a compactum K. The **open degree** *of K, if it exists, is the least positive integer n for which we can find a countable family $\mathcal{F}$ of open subsets of K such that for every one-to-one $(n+1)$-sequence $x_0, \ldots, x_n \in K$ there exist $V_0, \ldots, V_n \in \mathcal{F}$ such that:*

- $x_i \in V_i$ *for all* $i \leq n$,
- $\bigcap_0^n V_i = \emptyset$.

Put $\mathrm{odeg}(K) = \infty$ *if such n does not exist.*

Example

$\mathrm{odeg}(S(2^{\mathbb{N}})) = \mathrm{odeg}(D(2^{\mathbb{N}})) = \mathrm{odeg}(C(2^{<\mathbb{N}})) = 2.$

Open degrees

Definition

Fix a compactum K. The **open degree** *of K, if it exists, is the least positive integer n for which we can find a countable family $\mathcal{F}$ of open subsets of K such that for every one-to-one $(n+1)$-sequence $x_0, \dots, x_n \in K$ there exist $V_0, \dots, V_n \in \mathcal{F}$ such that:*

- $x_i \in V_i$ *for all* $i \leq n$,
- $\bigcap_0^n V_i = \emptyset$.

Put $\mathrm{odeg}(K) = \infty$ *if such n does not exist.*

Example

$\mathrm{odeg}(S(2^{\mathbb{N}})) = \mathrm{odeg}(D(2^{\mathbb{N}})) = \mathrm{odeg}(C(2^{<\mathbb{N}})) = 2.$

Proposition

$\mathrm{odeg}(K) = 1$ *iff K is metrizable.*

Co-zero degrees

Definition

For a compactum K, the **co-zero degree** *of K, if it exists, is the least positive integer n for which we can find a countable family $\mathcal{F}$ of open F_σ-subsets of K such that for every one-to-one $(n+1)$-sequence $x_0, \ldots, x_n \in K$ there exist $V_0, \ldots, V_n \in \mathcal{F}$ such that:*

- $x_i \in V_i$ *for all* $i \leq n$,
- $\bigcap_0^n V_i = \emptyset$.

Put $\mathrm{cozdeg}(K) = \infty$ *if such n does not exist.*

Co-zero degrees

Definition

For a compactum K, the **co-zero degree** *of K, if it exists, is the least positive integer n for which we can find a countable family $\mathcal{F}$ of open F_σ-subsets of K such that for every one-to-one $(n+1)$-sequence $x_0, \ldots, x_n \in K$ there exist $V_0, \ldots, V_n \in \mathcal{F}$ such that:*

- $x_i \in V_i$ *for all* $i \leq n$,
- $\bigcap_0^n V_i = \emptyset$.

Put $\mathrm{cozdeg}(K) = \infty$ *if such n does not exist.*

Example

$\mathrm{cozdeg}(S(2^{\mathbb{N}})) = \mathrm{cozdeg}(D(2^{\mathbb{N}})) = 2$ *but* $\mathrm{cozdeg}(C(2^{<\mathbb{N}})) = \infty$.

Proposition

$\mathrm{cozdeg}(K) \leq n$ *iff there is a continuous map* $f : K \to M$ *from* K *into some metric space* M *such that* $|f^{-1}(x)| \leq n$ *for all* $x \in M$.

Proposition

$\mathrm{cozdeg}(K) \leq n$ *iff there is a continuous map* $f : K \to M$ *from* K *into some metric space* M *such that* $|f^{-1}(x)| \leq n$ *for all* $x \in M$.

Theorem (T., 1999)

Let K *be a separable BC1-compactum. Then either*

- K *contains a* **discrete subspace of cardinality continuum**, *or*

Proposition

$\mathrm{cozdeg}(K) \leq n$ iff there is a continuous map $f : K \to M$ from K into some metric space M such that $|f^{-1}(x)| \leq n$ for all $x \in M$.

Theorem (T., 1999)

Let K be a separable BC1-compactum. Then either

- *K contains a* **discrete subspace of cardinality continuum**, *or*
- *there is a continuous map $f : K \to M$ from K into some metric space M such that $|f^{-1}(x)| \leq 2$ for all $x \in M$,*

Proposition

$\mathrm{cozdeg}(K) \leq n$ *iff there is a continuous map* $f : K \to M$ *from* K *into some metric space* M *such that* $|f^{-1}(x)| \leq n$ *for all* $x \in M$.

Theorem (T., 1999)

Let K *be a separable BC1-compactum. Then either*

- K *contains a* **discrete subspace of cardinality continuum**, *or*
- *there is a continuous map* $f : K \to M$ *from* K *into some metric space* M *such that* $|f^{-1}(x)| \leq 2$ *for all* $x \in M$, *i.e.,* $\mathrm{cozdeg}(K) \leq 2$.

An array of basis problems

An array of basis problems

Proposition

A BC1-compactum K is non-metrizable iff $\mathrm{odeg}(K) \geq 2$.

An array of basis problems

Proposition

A BC1-compactum K is non-metrizable iff $\mathrm{odeg}(K) \geq 2$.

Corollary

The class of separable BC1-compacta of open degree at least 2 has the 3-element basis

$$S(2^{\mathbb{N}}), D_{\mathrm{sep}}(2^{\mathbb{N}}), C(2^{<\mathbb{N}}).$$

An array of basis problems

Proposition

A BC1-compactum K is non-metrizable iff $\operatorname{odeg}(K) \geq 2$.

Corollary

The class of separable BC1-compacta of open degree at least 2 has the 3-element basis

$$S(2^{\mathbb{N}}), D_{\mathrm{sep}}(2^{\mathbb{N}}), C(2^{<\mathbb{N}}).$$

Question

Can a similar basis result be proved for other open degrees?

An array of basis problems

Proposition

A BC1-compactum K is non-metrizable iff $\operatorname{odeg}(K) \geq 2$.

Corollary

The class of separable BC1-compacta of open degree at least 2 has the 3-element basis

$$S(2^{\mathbb{N}}), D_{\mathrm{sep}}(2^{\mathbb{N}}), C(2^{<\mathbb{N}}).$$

Question

Can a similar basis result be proved for other open degrees?

Question

Are there any basis results for co-zero degrees?

A new finite basis theorem

A new finite basis theorem

Theorem (Aviles-T., 2015)

For every positive integer n, the class of BC1-compacta of open degree $\geq n$ has a finite basis that can be described explicitly.

A new finite basis theorem

Theorem (Aviles-T., 2015)

For every positive integer n*, the class of BC1-compacta of open degree* $\geq n$ *has a finite basis that can be described explicitly.*

Example

- *The class of BC1-compacta of open degree* ≥ 2 *has a 3-element basis.*

A new finite basis theorem

Theorem (Aviles-T., 2015)

For every positive integer n, the class of BC1-compacta of open degree $\geq n$ has a finite basis that can be described explicitly.

Example

- *The class of BC1-compacta of open degree ≥ 2 has a 3-element basis.*
- *The class of BC1-compacta of open degree ≥ 3 has a 4-element basis.*

A new finite basis theorem

Theorem (Aviles-T., 2015)

For every positive integer n, the class of BC1-compacta of open degree $\geq n$ has a finite basis that can be described explicitly.

Example

- *The class of BC1-compacta of open degree ≥ 2 has a 3-element basis.*
- *The class of BC1-compacta of open degree ≥ 3 has a 4-element basis.*
- *The class of BC1-compacta of open degree ≥ 4 has a 8-element basis.*

A new finite basis theorem

Theorem (Aviles-T., 2015)

For every positive integer n, the class of BC1-compacta of open degree $\geq n$ has a finite basis that can be described explicitly.

Example

- *The class of BC1-compacta of open degree ≥ 2 has a 3-element basis.*
- *The class of BC1-compacta of open degree ≥ 3 has a 4-element basis.*
- *The class of BC1-compacta of open degree ≥ 4 has a 8-element basis.*

Problem

Investigate the topological properties of the basic compacta and the corresponding classes of compacta they determine.

Some applications

Some applications

Theorem (Aviles-T., 2015)

Let K be a BC1-compactum. Then

$$K \text{ is scattered iff } S(2^{\mathbb{N}}) \not\to K \text{ and } 2^{\mathbb{N}} \not\to K.$$

Some applications

Theorem (Aviles-T., 2015)

Let K be a BC1-compactum. Then

$$K \text{ is scattered iff } S(2^{\mathbb{N}}) \not\to K \text{ and } 2^{\mathbb{N}} \not\to K.$$

Theorem (Aviles-T., 2015)

Suppose K is a BC1-compactum and that

$$f : K \to S(2^{\mathbb{N}})$$

is a continuous onto map. Then there is $K_0 \subseteq K$ homeomorphic to $S(2^{\mathbb{N}})$ such that

$$f \restriction K_0 \text{ is one-to-one.}$$

Details from the proof of the finite basis theorem

Details from the proof of the finite basis theorem

Definition

A topological space X is **bi-sequential** *if for every ultrafilter $\mathcal{U}$ on X converging to a point $x \in X$ there is a sequence A_n of elements of $\mathcal{U}$ converging to x.*

Details from the proof of the finite basis theorem

Definition

A topological space X is **bi-sequential** *if for every ultrafilter $\mathcal{U}$ on X converging to a point $x \in X$ there is a sequence A_n of elements of $\mathcal{U}$ converging to x.*

Theorem (Pol 1984, Debs 1987)

Every BC1-compactum is bi-sequential.

Details from the proof of the finite basis theorem

Definition

A topological space X is **bi-sequential** *if for every ultrafilter $\mathcal{U}$ on X converging to a point $x \in X$ there is a sequence A_n of elements of $\mathcal{U}$ converging to x.*

Theorem (Pol 1984, Debs 1987)

Every BC1-compactum is bi-sequential.

Corollary (Knaust 1991)

Every BC1-compactum has the **weak diagonal sequence property***, i.e., if x_n is a sequence of elements of K converging to a point $x \in K$ and if for every n, we have a sequence x_n^m converging to x_n then there is infinite $N \subseteq \omega$ and for each $n \in N$ an infinite set $M_n \subseteq \omega$ such that*

$$\{x_n^m : n \in N, m \in M_n\} \to x.$$

Extension Theorem

Extension Theorem

Theorem

Suppose that K_0 and K_1 are two bi-sequential spaces and that D_0 is a dense subset of K_0.

Extension Theorem

Theorem

Suppose that K_0 *and* K_1 *are two bi-sequential spaces and that* D_0 *is a dense subset of* K_0*. Suppose*

$$f : K_0 \to K_1$$

has the property that sequences in D_0 *that converge to the same point in* K_0 *are map to sequences that converge to the same pint in* K_1.

Extension Theorem

Theorem

Suppose that K_0 and K_1 are two bi-sequential spaces and that D_0 is a dense subset of K_0. Suppose

$$f : K_0 \to K_1$$

has the property that sequences in D_0 that converge to the same point in K_0 are map to sequences that converge to the same pint in K_1.
Then f extends to a continuous function

$$\bar{f} : K_0 \to K_1.$$

Trees and open degrees

Trees and open degrees

Lemma

Let K be a separable BC1-compactum of open degree $\geq m$ and let D be a countable dense subset of K. Then there is a one-to one mapping

$$f : m^{<\mathbb{N}} \to D$$

such that

$$\overline{\{f(t) : t \frown i \sqsubseteq z\}} \cap \overline{\{f(t) : t \frown j \sqsubseteq z\}} = \emptyset$$

for all $i < j < m$.

Trees and open degrees

Lemma

Let K be a separable BC1-compactum of open degree $\geq m$ and let D be a countable dense subset of K. Then there is a one-to one mapping

$$f : m^{<\mathbb{N}} \to D$$

such that

$$\overline{\{f(t) : t \frown i \sqsubseteq z\}} \cap \overline{\{f(t) : t \frown j \sqsubseteq z\}} = \emptyset$$

for all $i < j < m$.

Remark

So, in order to apply the Extension Theorem we need to:

Trees and open degrees

Lemma

Let K be a separable BC1-compactum of open degree $\geq m$ and let D be a countable dense subset of K. Then there is a one-to one mapping

$$f : m^{<\mathbb{N}} \to D$$

such that

$$\overline{\{f(t) : t \frown i \sqsubseteq z\}} \cap \overline{\{f(t) : t \frown j \sqsubseteq z\}} = \emptyset$$

for all $i < j < m$.

Remark

So, in order to apply the Extension Theorem we need to:

- *assign BC1-compacta to trees of the form $m^{<\mathbb{N}}$,*

Trees and open degrees

Lemma

Let K be a separable BC1-compactum of open degree $\geq m$ and let D be a countable dense subset of K. Then there is a one-to one mapping

$$f : m^{<\mathbb{N}} \to D$$

such that

$$\overline{\{f(t) : t \frown i \sqsubseteq z\}} \cap \overline{\{f(t) : t \frown j \sqsubseteq z\}} = \emptyset$$

for all $i < j < m$.

Remark

So, in order to apply the Extension Theorem we need to:

- *assign BC1-compacta to trees of the form $m^{<\mathbb{N}}$,*
- *develop the corresponding Ramsey-theory on trees.*

A Ramsey theorem for trees $m^{<\mathbb{N}}$

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A\rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A\rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

If $r = s \wedge t \notin \{s, t\}$ then $r^\frown i \sqsubseteq s$ and $r^\frown j \sqsubseteq t$ for distinct $i, j \in m$. In that case, we define **the incidence** as $inc(s, t) = (i, j)$.

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A\rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

If $r = s \wedge t \notin \{s, t\}$ then $r^\frown i \sqsubseteq s$ and $r^\frown j \sqsubseteq t$ for distinct $i, j \in m$. In that case, we define **the incidence** as $inc(s, t) = (i, j)$.

Definition

Sets $A, B \subset m^{<\omega}$ are **equivalent**, *$A \approx B$, if there is a bijection $f : \langle\langle A\rangle\rangle \to \langle\langle B\rangle\rangle$ such that for every $t, s \in \langle\langle A\rangle\rangle$*

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A\rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

If $r = s \wedge t \notin \{s, t\}$ then $r^\frown i \sqsubseteq s$ and $r^\frown j \sqsubseteq t$ for distinct $i, j \in m$. In that case, we define **the incidence** as $inc(s, t) = (i, j)$.

Definition

Sets $A, B \subset m^{<\omega}$ are **equivalent**, *$A \approx B$, if there is a bijection $f : \langle\langle A\rangle\rangle \to \langle\langle B\rangle\rangle$ such that for every $t, s \in \langle\langle A\rangle\rangle$*

- *$a \in A$ iff $f(a) \in B$*

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A \rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

If $r = s \wedge t \notin \{s, t\}$ then $r^\frown i \sqsubseteq s$ and $r^\frown j \sqsubseteq t$ for distinct $i, j \in m$. In that case, we define **the incidence** as $inc(s, t) = (i, j)$.

Definition

Sets $A, B \subset m^{<\omega}$ *are* **equivalent**, $A \approx B$, *if there is a bijection* $f : \langle\langle A \rangle\rangle \to \langle\langle B \rangle\rangle$ *such that for every* $t, s \in \langle\langle A \rangle\rangle$

- $a \in A$ *iff* $f(a) \in B$
- $f(t \wedge s) = f(t) \wedge f(s)$

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A\rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

If $r = s \wedge t \notin \{s, t\}$ then $r^\frown i \sqsubseteq s$ and $r^\frown j \sqsubseteq t$ for distinct $i, j \in m$. In that case, we define **the incidence** as $inc(s, t) = (i, j)$.

Definition

Sets $A, B \subset m^{<\omega}$ are **equivalent**, *$A \approx B$, if there is a bijection $f : \langle\langle A\rangle\rangle \to \langle\langle B\rangle\rangle$ such that for every $t, s \in \langle\langle A\rangle\rangle$*

- *$a \in A$ iff $f(a) \in B$*
- *$f(t \wedge s) = f(t) \wedge f(s)$*
- *$f(t) \prec f(s)$ if and only if $t \prec s$*

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A\rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

If $r = s \wedge t \notin \{s, t\}$ then $r^\frown i \sqsubseteq s$ and $r^\frown j \sqsubseteq t$ for distinct $i, j \in m$. In that case, we define **the incidence** as $inc(s, t) = (i, j)$.

Definition

Sets $A, B \subset m^{<\omega}$ are **equivalent**, *$A \approx B$, if there is a bijection $f : \langle\langle A\rangle\rangle \to \langle\langle B\rangle\rangle$ such that for every $t, s \in \langle\langle A\rangle\rangle$*

- *$a \in A$ iff $f(a) \in B$*
- *$f(t \wedge s) = f(t) \wedge f(s)$*
- *$f(t) \prec f(s)$ if and only if $t \prec s$*
- *If $i \in m$ is such that $t^\frown i \sqsubseteq s$, then $f(t)^\frown i \sqsubseteq f(s)$.*

A Ramsey theorem for trees $m^{<\mathbb{N}}$

Let $\sqsubseteq$ denote the usual end-extension ordering of the tree $m^{<\mathbb{N}}$ and $\prec$ its ω-ordering (first by the ordering of lenghts and then by the lexicographical ordering).

$\langle\langle A\rangle\rangle = \{s \wedge t : s, t \in A\}$ is the **meet-closure** of $A \subseteq m^{<\mathbb{N}}$.

If $r = s \wedge t \notin \{s, t\}$ then $r^\frown i \sqsubseteq s$ and $r^\frown j \sqsubseteq t$ for distinct $i, j \in m$. In that case, we define **the incidence** as $inc(s, t) = (i, j)$.

Definition

Sets $A, B \subset m^{<\omega}$ *are* **equivalent**, $A \approx B$, *if there is a bijection* $f : \langle\langle A\rangle\rangle \to \langle\langle B\rangle\rangle$ *such that for every* $t, s \in \langle\langle A\rangle\rangle$

- $a \in A$ *iff* $f(a) \in B$
- $f(t \wedge s) = f(t) \wedge f(s)$
- $f(t) \prec f(s)$ *if and only if* $t \prec s$
- *If* $i \in m$ *is such that* $t^\frown i \sqsubseteq s$*, then* $f(t)^\frown i \sqsubseteq f(s)$.

An (i, j)-**comb** is a subset $A \subseteq m^{<\omega}$ such that

$$A \approx \{(j), (iij), (iiiij), (iiiiiij), \ldots\}.$$

A basic Ramsey tool

A basic Ramsey tool

Theorem

Fix a set $A_0 \subseteq m^{<\mathbb{N}}$*, and a partition*

$$\{A \subseteq m^{<\mathbb{N}} : A \approx A_0\} = P_1 \cup \cdots \cup P_k$$

into finitely many sets with the property of Baire. Then there exists a subtree $T \subseteq m^{<\mathbb{N}}$ *equivalent to* $m^{<\mathbb{N}}$ *such that the family* $\{A \subseteq T : A \approx A_0\}$ *is contained in a single piece of the partition.*

A basic Ramsey tool

Theorem

Fix a set $A_0 \subseteq m^{<\mathbb{N}}$, and a partition

$$\{A \subseteq m^{<\mathbb{N}} : A \approx A_0\} = P_1 \cup \cdots \cup P_k$$

into finitely many sets with the property of Baire. Then there exists a subtree $T \subseteq m^{<\mathbb{N}}$ equivalent to $m^{<\mathbb{N}}$ such that the family $\{A \subseteq T : A \approx A_0\}$ is contained in a single piece of the partition.

Remark

So if there are BC1-compactifications of $m^{<\mathbb{N}}$ (taken with its discrete topology) in which all combs of the tree $m^{<\mathbb{N}}$ are convergent,

A basic Ramsey tool

Theorem

Fix a set $A_0 \subseteq m^{<\mathbb{N}}$*, and a partition*

$$\{A \subseteq m^{<\mathbb{N}} : A \approx A_0\} = P_1 \cup \cdots \cup P_k$$

into finitely many sets with the property of Baire. Then there exists a subtree $T \subseteq m^{<\mathbb{N}}$ *equivalent to* $m^{<\mathbb{N}}$ *such that the family* $\{A \subseteq T : A \approx A_0\}$ *is contained in a single piece of the partition.*

Remark

So if there are BC1-compactifications of $m^{<\mathbb{N}}$ *(taken with its discrete topology) in which all combs of the tree* $m^{<\mathbb{N}}$ *are convergent, we are halfway to proving the basis theorem.*

A basic Ramsey tool

Theorem

Fix a set $A_0 \subseteq m^{<\mathbb{N}}$, and a partition

$$\{A \subseteq m^{<\mathbb{N}} : A \approx A_0\} = P_1 \cup \cdots \cup P_k$$

into finitely many sets with the property of Baire. Then there exists a subtree $T \subseteq m^{<\mathbb{N}}$ equivalent to $m^{<\mathbb{N}}$ such that the family $\{A \subseteq T : A \approx A_0\}$ is contained in a single piece of the partition.

Remark

So if there are BC1-compactifications of $m^{<\mathbb{N}}$ (taken with its discrete topology) in which all combs of the tree $m^{<\mathbb{N}}$ are convergent, we are halfway to proving the basis theorem. The finite basis is to be found in the class of all such compactifications of $m^{<\mathbb{N}}$.

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

- $m^{<\mathbb{N}}$ is considered as a countable discrete space,

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

- $m^{<\mathbb{N}}$ is considered as a countable discrete space,
- $m^{\mathbb{N}}$ is considered with its product topology,

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

- $m^{<\mathbb{N}}$ is considered as a countable discrete space,
- $m^{\mathbb{N}}$ is considered with its product topology,
- $\mathfrak{P}$ is considered as a finite discrete space,

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

- $m^{<\mathbb{N}}$ is considered as a countable discrete space,
- $m^{\mathbb{N}}$ is considered with its product topology,
- $\mathfrak{P}$ is considered as a finite discrete space,
- $m^{\mathbb{N}} \times \mathfrak{P}$ is given the product topology,

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

- $m^{<\mathbb{N}}$ is considered as a countable discrete space,
- $m^{\mathbb{N}}$ is considered with its product topology,
- $\mathfrak{P}$ is considered as a finite discrete space,
- $m^{\mathbb{N}} \times \mathfrak{P}$ is given the product topology,
- $X_{\mathfrak{P}} = m^{<\mathbb{N}} \cup m^{\mathbb{N}} \times \mathfrak{P}$ is given the disjoint sum topology.

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

- $m^{<\mathbb{N}}$ is considered as a countable discrete space,
- $m^{\mathbb{N}}$ is considered with its product topology,
- $\mathfrak{P}$ is considered as a finite discrete space,
- $m^{\mathbb{N}} \times \mathfrak{P}$ is given the product topology,
- $X_{\mathfrak{P}} = m^{<\mathbb{N}} \cup m^{\mathbb{N}} \times \mathfrak{P}$ is given the disjoint sum topology.

For $s \in m^{<\mathbb{N}}$ let $\mathbf{f}_s : X_{\mathfrak{P}} \to \{0, 1\}$ be given by

BC1-compactifications of $m^{<\mathbb{N}}$

Given a partition $\mathfrak{P}$ of $m \times m$, define the Polish space $X_{\mathfrak{P}} := m^{<\omega} \cup m^{\omega} \times \mathfrak{P}$ as follows:

- $m^{<\mathbb{N}}$ is considered as a countable discrete space,
- $m^{\mathbb{N}}$ is considered with its product topology,
- $\mathfrak{P}$ is considered as a finite discrete space,
- $m^{\mathbb{N}} \times \mathfrak{P}$ is given the product topology,
- $X_{\mathfrak{P}} = m^{<\mathbb{N}} \cup m^{\mathbb{N}} \times \mathfrak{P}$ is given the disjoint sum topology.

For $s \in m^{<\mathbb{N}}$ let $\mathbf{f}_s : X_{\mathfrak{P}} \to \{0, 1\}$ be given by

$$\mathbf{f}_s(t) = \begin{cases} 1 \text{ if } t \leq s \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } t \in m^{<\omega},$$

$$\mathbf{f}_s(y, Q) = \begin{cases} 1 \text{ if } inc(y, s) \in Q \\ 0 \text{ if } inc(y, s) \notin Q \end{cases} \qquad \text{for } (y, Q) \in m^{\omega} \times \mathfrak{P}$$

Definition

The compact space $K_1(\mathfrak{P})$ *is the pointwise closure of* $\{\mathbf{f}_s : s \in m^{<\mathbb{N}}\}$ *in* $\{0,1\}^{X_{\mathfrak{P}}}$.

Definition

The compact space $K_1(\mathfrak{P})$ *is the pointwise closure of* $\{\mathbf{f}_s : s \in m^{<\mathbb{N}}\}$ *in* $\{0,1\}^{X_{\mathfrak{P}}}$*.*

To describe the points of $K_1(\mathfrak{P})$, for every $(x, P) \in m^{\mathbb{N}} \times \mathfrak{P}$, we attach a function $\mathbf{f}_{(x,P)} : X_{\mathfrak{P}} \to \{0,1\}$ given by

Definition

The compact space $K_1(\mathfrak{P})$ is the pointwise closure of $\{\mathbf{f}_s : s \in m^{<\mathbb{N}}\}$ in $\{0,1\}^{X_{\mathfrak{P}}}$.

To describe the points of $K_1(\mathfrak{P})$, for every $(x, P) \in m^{\mathbb{N}} \times \mathfrak{P}$, we attach a function $\mathbf{f}_{(x,P)} : X_{\mathfrak{P}} \to \{0,1\}$ given by

$$\mathbf{f}_{(x,P)}(t) = \begin{cases} 1 \text{ if } t \leq x \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } t \in m^{<\mathbb{N}}$$

$$\mathbf{f}_{(x,P)}(y, Q) = \begin{cases} 1 \text{ if } x = y,\ \ P = Q \\ 0 \text{ if } x = y,\ \ P \neq Q \\ 1 \text{ if } x \neq y, inc(y, x) \in Q \\ 0 \text{ if } x \neq y, inc(y, x) \notin Q \end{cases} \qquad \text{for } (y, Q) \in m^{\mathbb{N}} \times \mathfrak{P}$$

Proposition

For $(i, j) \in P \in \mathfrak{P}$:

Proposition

For $(i,j) \in P \in \mathfrak{P}$:

- *If* $\{s_0, s_1, \ldots\} \subset m^{<\omega}$ *is an* (i,j)-**sequence over** $x \in m^{\mathbb{N}}$ *(i.e., enumeration of an* (i,j)*-comb converging to* x*), then*

$$\lim_k \mathbf{f}_{s_k} = \mathbf{f}_{(x,P)}.$$

Proposition

For $(i, j) \in P \in \mathfrak{P}$:

- *If* $\{s_0, s_1, \ldots\} \subset m^{<\omega}$ *is an* (i, j)-**sequence over** $x \in m^{\mathbb{N}}$ *(i.e., enumeration of an* (i, j)*-comb converging to* x*), then*

$$\lim_k \mathbf{f}_{s_k} = \mathbf{f}_{(x,P)}.$$

- *If* $\{x_0, x_1, \ldots\} \subset m^{\omega}$ *is an* (i, j)*-sequence over* $x \in m^{\mathbb{N}}$*, and we choose any* $P_k \in \mathfrak{P}$*, then*

$$\lim_k \mathbf{f}_{(x_k,P_k)} = \mathbf{f}_{(x,P)}.$$

Proposition

For $(i,j) \in P \in \mathfrak{P}$ *:*

- *If* $\{s_0, s_1, \ldots\} \subset m^{<\omega}$ *is an* (i,j)**-sequence over** $x \in m^{\mathbb{N}}$ *(i.e., enumeration of an* (i,j)*-comb converging to* x*), then*

$$\lim_k \mathbf{f}_{s_k} = \mathbf{f}_{(x,P)}.$$

- *If* $\{x_0, x_1, \ldots\} \subset m^{\omega}$ *is an* (i,j)*-sequence over* $x \in m^{\mathbb{N}}$*, and we choose any* $P_k \in \mathfrak{P}$*, then*

$$\lim_k \mathbf{f}_{(x_k,P_k)} = \mathbf{f}_{(x,P)}.$$

Proposition

$$K_1(\mathfrak{P}) = \{\mathbf{f}_s : s \in m^{<\mathbb{N}}\} \cup \{\mathbf{f}_{(x,P)} : (x,P) \in m^{\mathbb{N}} \times \mathfrak{P}\}.$$

The points $\mathbf{f}_s$ *are isolated and the points* $\mathbf{f}_{(x.P)}$ *are* G_δ*-points, so* $K_1(\mathfrak{P})$ *is a* **first-countable space.**

Proposition

If the subtree $T \subseteq m^{<\mathbb{N}}$ *is equivalent to* $m^{<\mathbb{N}}$*, then the closure of* $\{f_t : t \in T\}$ *is naturally homeomorphic to the whole space* $K_1(\mathfrak{P})$*.*

Proposition

If the subtree $T \subseteq m^{<\mathbb{N}}$ *is equivalent to* $m^{<\mathbb{N}}$*, then the closure of* $\{f_t : t \in T\}$ *is naturally homeomorphic to the whole space* $K_1(\mathfrak{P})$*.*

Example

When $m = 2$*, we have the following two natural partitions of* 2×2 *and the corresponding separable BC1-compacta:*

Proposition

If the subtree $T \subseteq m^{<\mathbb{N}}$ *is equivalent to* $m^{<\mathbb{N}}$*, then the closure of* $\{f_t : t \in T\}$ *is naturally homeomorphic to the whole space* $K_1(\mathfrak{P})$*.*

Example

When $m = 2$*, we have the following two natural partitions of* 2×2 *and the corresponding separable BC1-compacta:*

- *Let* $\mathfrak{P}_2^0 = \{\{(0,0),(1,1),(1,0)\},\{(0,1)\}\}$ *Then the space* $K_1(\mathfrak{P}_2^0)$ *both contains and is contained in the* **split interval***.*

Proposition

If the subtree $T \subseteq m^{<\mathbb{N}}$ is equivalent to $m^{<\mathbb{N}}$, then the closure of $\{f_t : t \in T\}$ is naturally homeomorphic to the whole space $K_1(\mathfrak{P})$.

Example

When $m = 2$, we have the following two natural partitions of 2×2 and the corresponding separable BC1-compacta:

- *Let $\mathfrak{P}_2^0 = \{\{(0,0),(1,1),(1,0)\},\{(0,1)\}\}$ Then the space $K_1(\mathfrak{P}_2^0)$ both contains and is contained in the* **split interval**.
- *Let $\mathfrak{P}_2^1 = \{\{(0,0)\},\{(0,1),(1,0),(1,1)\}\}$. Then*

$$\{\mathbf{f}_{(x,P)} : x \in 2^\omega, P \in \mathfrak{P}_2^1\}$$

is homeomorphic to the **Alexandrov duplicate** *of the Cantor set and $K_1(\mathfrak{P}_2^1)$ is its* **separable extension.**

Recognizing classical spaces

Recognizing classical spaces

Lemma

$K_1(\mathfrak{P})$ contains a homeomorphic copy of the Cantor set if and only if there exist $i \neq j$ such that (i, j) and (j, i) live in the same piece of the partition.

Recognizing classical spaces

Lemma

$K_1(\mathfrak{P})$ contains a homeomorphic copy of the Cantor set if and only if there exist $i \neq j$ such that (i,j) and (j,i) live in the same piece of the partition.

Lemma

If $g : \{0,1\}^2 \to \{0,1\}$ is such that $g(0,1) \neq g(1,0)$ and $\mathfrak{P}_g \neq \{\{(0,0),(1,0)\},\{(1,1),(0,1)\}\}$, then $K_1(\mathfrak{P}_g)$ is homeomorphic to a subspace of the split interval.

Recognizing classical spaces

Lemma

$K_1(\mathfrak{P})$ contains a homeomorphic copy of the Cantor set if and only if there exist $i \neq j$ such that (i,j) and (j,i) live in the same piece of the partition.

Lemma

If $g : \{0,1\}^2 \to \{0,1\}$ is such that $g(0,1) \neq g(1,0)$ and $\mathfrak{P}_g \neq \{\{(0,0),(1,0)\},\{(1,1),(0,1)\}\}$, then $K_1(\mathfrak{P}_g)$ is homeomorphic to a subspace of the split interval.

Lemma

$K_1(\mathfrak{P})$ contains a homeomorphic copy of the split interval if and only if there exist $i \neq j$ such that (i,j) and (j,i) live in different pieces of the partition $\mathfrak{P}$.

Recognizing classical spaces

Lemma

$K_1(\mathfrak{P})$ contains a homeomorphic copy of the Cantor set if and only if there exist $i \neq j$ such that (i,j) and (j,i) live in the same piece of the partition.

Lemma

If $g : \{0,1\}^2 \to \{0,1\}$ is such that $g(0,1) \neq g(1,0)$ and $\mathfrak{P}_g \neq \{\{(0,0),(1,0)\},\{(1,1),(0,1)\}\}$, then $K_1(\mathfrak{P}_g)$ is homeomorphic to a subspace of the split interval.

Lemma

$K_1(\mathfrak{P})$ contains a homeomorphic copy of the split interval if and only if there exist $i \neq j$ such that (i,j) and (j,i) live in different pieces of the partition $\mathfrak{P}$.

Theorem

Let K be a Rosenthal compact space that is **not scattered.** *Then K contains either a homeomorphic copy of the* **Cantor set** *or a homeomorphic copy of the* **split interval.**

Non first countable examples

Non first countable examples

Fix a family $\mathfrak{Q}$ of disjoint subsets of $m = \{0, 1, \ldots, m-1\}$ and let

$$X_{\mathfrak{Q}} := m^{<\mathbb{N}} \cup m^{\mathbb{N}} \times \mathfrak{Q}$$

be the corresponding Polish space.

Non first countable examples

Fix a family $\mathfrak{Q}$ of disjoint subsets of $m = \{0, 1, \ldots, m-1\}$ and let

$$X_{\mathfrak{Q}} := m^{<\mathbb{N}} \cup m^{\mathbb{N}} \times \mathfrak{Q}$$

be the corresponding Polish space.
For $s \in m^{<\omega}$, let $\mathbf{g}_s : X_{\mathfrak{Q}} \to \{0, 1\}$ be given by

$$\mathbf{g}_s(t) = \begin{cases} 1 \text{ if } t = s \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } t \in m^{<\omega},$$

$$\mathbf{g}_s(y, Q) = \begin{cases} 1 \text{ if } inc(y, s) = (i, i) \text{ for some } i \in Q \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } (y, Q) \in m^{\omega} \times \mathfrak{Q}$$

Non first countable examples

Fix a family $\mathfrak{Q}$ of disjoint subsets of $m = \{0, 1, \ldots, m-1\}$ and let

$$X_{\mathfrak{Q}} := m^{<\mathbb{N}} \cup m^{\mathbb{N}} \times \mathfrak{Q}$$

be the corresponding Polish space.
For $s \in m^{<\omega}$, let $\mathbf{g}_s : X_{\mathfrak{Q}} \to \{0, 1\}$ be given by

$$\mathbf{g}_s(t) = \begin{cases} 1 \text{ if } t = s \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } t \in m^{<\omega},$$

$$\mathbf{g}_s(y, Q) = \begin{cases} 1 \text{ if } inc(y, s) = (i, i) \text{ for some } i \in Q \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } (y, Q) \in m^{\omega} \times \mathfrak{Q}$$

Definition

The compact space $K_{\infty}(\mathfrak{Q})$ is the pointwise closure of the functions $\{\mathbf{g}_s : s \in m^{<\mathbb{N}}\}$ in $\{0, 1\}^{X_{\mathfrak{Q}}}$.

Let $\mathbf{g}_\infty : X_{\mathfrak{Q}} \to \{0, 1\}$ be constantly equal to 0 function.

Let $\mathbf{g}_\infty : X_{\mathfrak{Q}} \to \{0,1\}$ be constantly equal to 0 function.
For $(x, P) \in m^\omega \times \mathfrak{Q}$, let

$$\mathbf{g}_{(x,P)} : X_{\mathfrak{Q}} \to \{0,1\}$$

is 0 at all points except at (x, P), where it takes value 1.

Let $\mathbf{g}_\infty : X_{\mathfrak{Q}} \to \{0,1\}$ be constantly equal to 0 function.
For $(x, P) \in m^\omega \times \mathfrak{Q}$, let

$$\mathbf{g}_{(x,P)} : X_{\mathfrak{Q}} \to \{0,1\}$$

is 0 at all points except at (x, P), where it takes value 1.

Proposition

Fix $i, j \in m$, and $\{s_0, s_1, \ldots\} \subset m^{<\mathbb{N}}$ an (i,j)-sequence over $x \in m^{\mathbb{N}}$.

Let $\mathbf{g}_\infty : X_{\mathfrak{Q}} \to \{0,1\}$ be constantly equal to 0 function.
For $(x, P) \in m^\omega \times \mathfrak{Q}$, let

$$\mathbf{g}_{(x,P)} : X_{\mathfrak{Q}} \to \{0,1\}$$

is 0 at all points except at (x, P), where it takes value 1.

Proposition

Fix $i, j \in m$, and $\{s_0, s_1, \dots\} \subset m^{<\mathbb{N}}$ an (i,j)-sequence over $x \in m^{\mathbb{N}}$.

- *If $i = j \in P \in \mathfrak{Q}$, then $\lim_k \mathbf{g}_{s_k} = \mathbf{g}_{(x,P)}$.*

Let $\mathbf{g}_\infty : X_{\mathfrak{Q}} \to \{0,1\}$ be constantly equal to 0 function.
For $(x,P) \in m^\omega \times \mathfrak{Q}$, let

$$\mathbf{g}_{(x,P)} : X_{\mathfrak{Q}} \to \{0,1\}$$

is 0 at all points except at (x,P), where it takes value 1.

Proposition

Fix $i,j \in m$, and $\{s_0, s_1, \ldots\} \subset m^{<\mathbb{N}}$ an (i,j)-sequence over $x \in m^{\mathbb{N}}$.

- *If $i = j \in P \in \mathfrak{Q}$, then $\lim_k \mathbf{g}_{s_k} = \mathbf{g}_{(x,P)}$.*
- *If either $i \neq j$ or $i = j \notin \bigcup \mathfrak{Q}$, then $\lim_k \mathbf{g}_{s_k} = \mathbf{g}_\infty$.*

On the other hand, the only accumulation point of the set $\{\mathbf{g}_{(x,P)} : x \in m^{\mathbb{N}}, P \in \mathfrak{Q}\}$ is $\mathbf{g}_\infty$.

Let $\mathbf{g}_\infty : X_{\mathfrak{Q}} \to \{0,1\}$ be constantly equal to 0 function.
For $(x, P) \in m^\omega \times \mathfrak{Q}$, let

$$\mathbf{g}_{(x,P)} : X_{\mathfrak{Q}} \to \{0,1\}$$

is 0 at all points except at (x, P), where it takes value 1.

Proposition

Fix $i, j \in m$, and $\{s_0, s_1, \ldots\} \subset m^{<\mathbb{N}}$ an (i,j)-sequence over $x \in m^{\mathbb{N}}$.

- *If $i = j \in P \in \mathfrak{Q}$, then $\lim_k \mathbf{g}_{s_k} = \mathbf{g}_{(x,P)}$.*
- *If either $i \neq j$ or $i = j \notin \bigcup \mathfrak{Q}$, then $\lim_k \mathbf{g}_{s_k} = \mathbf{g}_\infty$.*

On the other hand, the only accumulation point of the set $\{\mathbf{g}_{(x,P)} : x \in m^{\mathbb{N}}, P \in \mathfrak{Q}\}$ is $\mathbf{g}_\infty$.

Corollary

$K_\infty(\mathfrak{Q})$ is a separable BC1-compactum.

Topological description of $K_\infty(\mathfrak{Q})$

Topological description of $K_\infty(\mathfrak{Q})$

Proposition

The function $X_{\mathfrak{Q}} \cup \{\infty\} \to K_\infty(\mathfrak{Q})$ given by $\xi \mapsto \mathbf{g}_\xi$ is a bijection. Thus,

$$K_\infty(\mathfrak{P}) = \{\mathbf{g}_s : s \in m^{<\omega}\} \cup \{\mathbf{g}_{(x,P)} : (x, P) \in m^{\mathbb{N}} \times \mathfrak{Q}\} \cup \{\mathbf{g}_\infty\}.$$

This is a scattered space of height 3, whose Cantor-Bendixson derivates are

$$\begin{aligned} K_\infty(\mathfrak{P})' &= \{\mathbf{g}_{(x,P)} : (x, P) \in m^{\mathbb{N}} \times \mathfrak{Q}\} \cup \{\mathbf{g}_\infty\} \\ K_\infty(\mathfrak{P})'' &= \{\mathbf{g}_\infty\} \end{aligned}$$

Thus, the points $\mathbf{g}_s$ are isolated in $K_\infty(\mathfrak{Q})$, the points $\mathbf{g}_{(x.P)}$ are G_δ-points, but if $\mathfrak{Q} \neq \emptyset$, then $\mathbf{g}_\infty$ is not a G_δ-point of $K_\infty(\mathfrak{Q})$.

Topological description of $K_\infty(\mathfrak{Q})$

Proposition

The function $X_{\mathfrak{Q}} \cup \{\infty\} \to K_\infty(\mathfrak{Q})$ given by $\xi \mapsto \mathbf{g}_\xi$ is a bijection. Thus,

$$K_\infty(\mathfrak{P}) = \{\mathbf{g}_s : s \in m^{<\omega}\} \cup \{\mathbf{g}_{(x,P)} : (x, P) \in m^{\mathbb{N}} \times \mathfrak{Q}\} \cup \{\mathbf{g}_\infty\}.$$

This is a scattered space of height 3, whose Cantor-Bendixson derivates are

$$\begin{aligned} K_\infty(\mathfrak{P})' &= \{\mathbf{g}_{(x,P)} : (x, P) \in m^{\mathbb{N}} \times \mathfrak{Q}\} \cup \{\mathbf{g}_\infty\} \\ K_\infty(\mathfrak{P})'' &= \{\mathbf{g}_\infty\} \end{aligned}$$

Thus, the points $\mathbf{g}_s$ are isolated in $K_\infty(\mathfrak{Q})$, the points $\mathbf{g}_{(x.P)}$ are G_δ-points, but if $\mathfrak{Q} \neq \emptyset$, then $\mathbf{g}_\infty$ is not a G_δ-point of $K_\infty(\mathfrak{Q})$.

Example

Let $m = 2$ and $\mathfrak{D}_2 = \{\{0, 1\}\}$. Then $K_\infty(\mathfrak{D}_2)$ is homeomorphic to the **Cantor tree compactum** *$C(2^{<\mathbb{N}})$.*

Comparing the two open degrees

Comparing the two open degrees

Theorem

$\mathrm{odeg}(K_1(\mathfrak{P})) = |\mathfrak{P}|$ *and* $\mathrm{odeg}(K_\infty(\mathfrak{Q})) = |\mathfrak{Q}| + 1$.

Comparing the two open degrees

Theorem

$\operatorname{odeg}(K_1(\mathfrak{P})) = |\mathfrak{P}|$ *and* $\operatorname{odeg}(K_\infty(\mathfrak{Q})) = |\mathfrak{Q}| + 1$.

Corollary

$\operatorname{odeg}(S(2^{\mathbb{N}})) = \operatorname{odeg}(D(2^{\mathbb{N}})) = 2$ *and* $\operatorname{odeg}(C(2^{<\mathbb{N}})) = 1$.

Comparing the two open degrees

Theorem

$\mathrm{odeg}(K_1(\mathfrak{P})) = |\mathfrak{P}|$ *and* $\mathrm{odeg}(K_\infty(\mathfrak{Q})) = |\mathfrak{Q}| + 1$.

Corollary

$\mathrm{odeg}(S(2^{\mathbb{N}})) = \mathrm{odeg}(D(2^{\mathbb{N}})) = 2$ *and* $\mathrm{odeg}(C(2^{<\mathbb{N}})) = 1$.

Proposition

$\mathrm{cozdeg}(C(2^{<\mathbb{N}})) = \infty$.

Comparing the two open degrees

Theorem

$\mathrm{odeg}(K_1(\mathfrak{P})) = |\mathfrak{P}|$ *and* $\mathrm{odeg}(K_\infty(\mathfrak{Q})) = |\mathfrak{Q}| + 1$.

Corollary

$\mathrm{odeg}(S(2^{\mathbb{N}})) = \mathrm{odeg}(D(2^{\mathbb{N}})) = 2$ *and* $\mathrm{odeg}(C(2^{<\mathbb{N}})) = 1$.

Proposition

$\mathrm{cozdeg}(C(2^{<\mathbb{N}})) = \infty$.

Proposition

For every positive integer m there is a **first countable** *BC1-compactum K such that* $\mathrm{odeg}(K) = 2$ *but* $\mathrm{cozdeg}(K) = m$.

Pre-metric compacta of degree n

Pre-metric compacta of degree n

Proposition

$\mathrm{cozdeg}(K) \le n$ iff there is a continuous map $f : K \to M$ into some metric space M such that $|f^{-1}(x)| \le n$ for all $x \in M$.

Pre-metric compacta of degree n

Proposition

$\mathrm{cozdeg}(K) \leq n$ iff there is a continuous map $f : K \to M$ into some metric space M such that $|f^{-1}(x)| \leq n$ for all $x \in M$.

Theorem (T., 1999)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq 2$. Then at least one of the following three conditions must hold:

Pre-metric compacta of degree n

Proposition

$\mathrm{cozdeg}(K) \leq n$ iff there is a continuous map $f : K \to M$ into some metric space M such that $|f^{-1}(x)| \leq n$ for all $x \in M$.

Theorem (T., 1999)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq 2$. Then at least one of the following three conditions must hold:

- *K is metrizable.*

Pre-metric compacta of degree n

Proposition

$\mathrm{cozdeg}(K) \leq n$ iff there is a continuous map $f : K \to M$ into some metric space M such that $|f^{-1}(x)| \leq n$ for all $x \in M$.

Theorem (T., 1999)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq 2$. Then at least one of the following three conditions must hold:

- *K is metrizable.*
- *K contains a homeomorphic copy of $S(2^{\mathbb{N}})$.*

Pre-metric compacta of degree n

Proposition

$\mathrm{cozdeg}(K) \leq n$ iff there is a continuous map $f : K \to M$ into some metric space M such that $|f^{-1}(x)| \leq n$ for all $x \in M$.

Theorem (T., 1999)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq 2$. Then at least one of the following three conditions must hold:

- *K is metrizable.*
- *K contains a homeomorphic copy of $S(2^{\mathbb{N}})$.*
- *K contains a homeomorphic copy of $D(2^{\mathbb{N}})$.*

Pre-metric compacta of degree n

Proposition

$\mathrm{cozdeg}(K) \leq n$ iff there is a continuous map $f : K \to M$ into some metric space M such that $|f^{-1}(x)| \leq n$ for all $x \in M$.

Theorem (T., 1999)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq 2$. Then at least one of the following three conditions must hold:

- *K is metrizable.*
- *K contains a homeomorphic copy of $S(2^{\mathbb{N}})$.*
- *K contains a homeomorphic copy of $D(2^{\mathbb{N}})$.*

Question

Is there a similar basis result for BC1-compacta K such that $\mathrm{cozdeg}(K) \leq n$?

The *n*-split interval

The n-split interval

Definition

Given a perfect subset P of the unit interval I and an integer $n \geq 2$, let $S_n(P)$ be the set $P \times \{0, 1, ..., n-1\}$ with the topology where the points of $P \times \{2, 3, ..., n-1\}$ are isolated and where the neighbourhoods of points $(x, 0)$ and $(x, 1)$ have respectively the following forms:

$$](y,1),(x,0)] \cup]y,x[\times\{2,3,...,n-1\} \text{ for } y < x,$$

$$[(x,1),(y,0)[\cup]x,y[\times\{2,3,...,n-1\}) \text{ for } y > x.$$

The n-split interval

Definition

Given a perfect subset P of the unit interval I and an integer $n \geq 2$, let $S_n(P)$ be the set $P \times \{0, 1, ..., n-1\}$ with the topology where the points of $P \times \{2, 3, ..., n-1\}$ are isolated and where the neighbourhoods of points $(x, 0)$ and $(x, 1)$ have respectively the following forms:

$$](y,1),(x,0)] \cup]y,x[\times\{2,3,...,n-1\} \text{ for } y < x,$$

$$[(x,1),(y,0)[\cup]x,y[\times\{2,3,...,n-1\}) \text{ for } y > x.$$

Proposition

For every integer $n \geq 2$, the space $S_n(I)$ is a BC1-compactum such that $\operatorname{cozdeg}(S_n(I)) = n$

The n-plicate

Definition

For a given topological space Z and integer $n \geq 2$, by $D_n(Z)$ we denote the space on $Z \times \{0, 1, ..., n-1\}$ in which all points of $Z \times \{1, 2, ..., n-1\}$ are isolated and the neighbourhoods of points $(z, 0)$ have the form

$$U \times \{0, 1, ..., n-1\} \setminus \{z\} \times \{0, 1, ..., n-1\},$$

where U is an arbitrary neighbourhood of z in Z.

The n-plicate

Definition

For a given topological space Z and integer $n \geq 2$, by $D_n(Z)$ we denote the space on $Z \times \{0, 1, ..., n-1\}$ in which all points of $Z \times \{1, 2, ..., n-1\}$ are isolated and the neighbourhoods of points $(z, 0)$ have the form

$$U \times \{0, 1, ..., n-1\} \setminus \{z\} \times \{0, 1, ..., n-1\},$$

where U is an arbitrary neighbourhood of z in Z.

Proposition

For every integer $n \geq 2$, the space $D_n(2^{\mathbb{N}})$ is a BC1-compactum such that $\mathrm{cozdeg}(D_n(2^{\mathbb{N}})) = n$.

Basis result for the co-zero degree

Basis result for the co-zero degree

Theorem (Aviles-Poveda-T., 2015)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq n$ *for some integer* $n \geq 2$.

Basis result for the co-zero degree

Theorem (Aviles-Poveda-T., 2015)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq n$ for some integer $n \geq 2$.
Then at aleast one of the following conditions must hold:

Basis result for the co-zero degree

Theorem (Aviles-Poveda-T., 2015)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq n$ for some integer $n \geq 2$.
Then at aleast one of the following conditions must hold:

- $\mathrm{cozdeg}(K) \leq n$.

Basis result for the co-zero degree

Theorem (Aviles-Poveda-T., 2015)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq n$ for some integer $n \geq 2$.
Then at aleast one of the following conditions must hold:

- $\mathrm{cozdeg}(K) \leq n$.
- *K contains a topological copy of $S_n(2^{\mathbb{N}})$.*

Basis result for the co-zero degree

Theorem (Aviles-Poveda-T., 2015)

Let K be a separable BC1-compactum such that $\mathrm{cozdeg}(K) \leq n$ for some integer $n \geq 2$.
Then at aleast one of the following conditions must hold:

- $\mathrm{cozdeg}(K) \leq n$.
- *K contains a topological copy of $S_n(2^{\mathbb{N}})$.*
- *K contains a topological copy of $D_n(2^{\mathbb{N}})$.*