Lindelöf number of compacta under the G_δ topology

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Theorem (Arhangel'skii, 1969)

For any T_2 space X, $|X| \leq 2^{\chi(X)L(X)}$

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- And many others...

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What can still be said about the class of compact T_2 spaces?

Compact spaces in the G_δ topology

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REMARK:
If X has countable pseudo-character, then X_δ is discrete.

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REMARK:
If X has countable pseudo-character, then X_δ is discrete.
Hence $|X| = wL(X) = L(X)$.

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The G_δ topology on $(\omega + 1)^\kappa$ has Lindelof degree κ (for $\kappa <$ the first inaccessible).

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Theorem (Gorelic, 1996)

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- X is ccc (Juhasz, 1972)
- Game theoretic property generalizing the ccc (Spadaro).

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Remark $X = [0,1]^{\mathfrak{c}^+}$ is ccc. So, by Juhasz's result, the weak Lindelöf degree of X_δ is continuum.

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  Hence does not cover any f such that $f \restriction \alpha$ is 1-1.

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$$\left(K_{\alpha+1} \setminus \overline{\bigcup \mathcal{U}_\alpha}\right) \neq \emptyset$$

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Question

Is there a compact T_2 space that is partitionable into more than continuum many G_δ sets? Is there a bound for the size of such partitions?

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Problem (van Douwen)

Do all compact homogeneous spaces have cellularity bound by $\mathfrak{c}$?

THANK YOU!