Baire classes of affine vector-valued functions

Ondřej Kalenda and Jiří Spurný

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*$\mu\in\mathcal{M}^+(X)$ is **maximal** if it is $\prec$-maximal.*

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For each $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$.

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Example

If K is a compact space, then $X = \mathcal{M}^1(K)$ is a simplex.

Harmonic functions

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Then

$$X = \{x^* \in \mathcal{H}^* \colon x^* \geq 0, \|x^*\| = 1\}$$

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$$(\mathcal{F})_\alpha = \{f\colon K \to L;\ \text{there exists a sequence } (f_n) \text{ in } \bigcup_{\beta<\alpha} (\mathcal{F})_\beta \text{ such that } f_n \to f \text{ pointwise}\}.$$

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- *If K and L are topological spaces, by $\mathcal{C}_\alpha(K, L)$ we denote the set $(\mathcal{C}(K, L))_\alpha$, where $\mathcal{C}(K, L)$ is the set of all continuous functions from K to L.*

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- *If K and L are topological spaces, by $\mathcal{C}_\alpha(K, L)$ we denote the set $(\mathcal{C}(K, L))_\alpha$, where $\mathcal{C}(K, L)$ is the set of all continuous functions from K to L.*
- *If X is a compact convex set and L is a convex subset of a locally convex space, by $\mathfrak{A}_\alpha(X, L)$ we denote $(\mathfrak{A}(X, L))_\alpha$, where $\mathfrak{A}(X, L)$ is the set of all affine continuous functions defined on X with values in L.*

If $(X, \mathcal{A}, \mu)$ is a measure space with μ finite, F a locally convex space and $f\colon X \to F$,

Vector integration (Pettis approach)

*If $(X, \mathcal{A}, \mu)$ is a measure space with μ finite, F a locally convex space and $f\colon X \to F$, then f is μ-**integrable** if*

- $\tau \circ f \in L^1(\mu)$ *for each* $\tau \in F^*$,
- *for each* $B \subset A$ μ*-measurable there exists an element* $x_B \in F$ *such that*

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Lemma

If K is a compact space, $\mu \in \mathcal{M}^+(K)$, F a Fréchet space and $f\colon K \to F$ bounded Baire measurable, then f is μ-integrable.

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***Baire sets** is the σ-algebra generated by **cozero** sets.*

Strongly affine functions

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If f is strongly affine, then f is affine and bounded.

Strongly affine functions of the first Baire class

Theorem (Choquet, Mokobodzki)

$f \in \mathcal{C}_1(X, \mathbb{R})$ affine, then f is strongly affine and in $\mathfrak{A}_1(X, \mathbb{R})$.

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If F is a locally convex space, then any affine $f \in \mathcal{C}_1(X, F)$ is strongly affine.

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If F is a Banach space with a bounded approximation property. Then any affine $f \in \mathcal{C}_1(X,F)$ is in $\mathfrak{A}_1(X,F)$.

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Example

If E is separable reflexive Banach space without the compact approximation property, $X = (B_E, w)$ and $f : X \to E$ is identity, then $f \in \mathcal{C}_1(X, F) \setminus \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X, F)$.

Strongly affine scalar functions of higher classes

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Theorem (Talagrand)

There exists a compact convex set X and a strongly affine function $f \in \mathcal{C}_2(X,\mathbb{R})$ such that $f \notin \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X,\mathbb{R})$.

Dilation mapping on simplices

Theorem

Let X be a simplex. Then the mapping

$$T\colon X \to \mathcal{M}^1(X), \quad x \mapsto \delta_x,$$

is strongly affine and in $\mathfrak{A}_1(X, \mathcal{M}^1(X))$.

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δ_x *is the unique maximal measure with* $r(\delta_x) = x$.

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Let X be a simplex, F be a Fréchet space, $1 \leq \alpha < \omega_1$ and $f \in \mathcal{C}_\alpha(X, F)$ be strongly affine. Then $f \in \mathfrak{A}_{1+\alpha}(X, F)$.

The Dirichlet problem on simplices

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Let X be a simplex with $\operatorname{ext} X$ being Lindelöf, $\alpha \in [0, \omega_1)$, F a Fréchet space and $f : \operatorname{ext} X \to F$ a bounded mapping from $\mathcal{C}_\alpha(\operatorname{ext} X, F)$.

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Let X be a simplex, $K \subset \operatorname{ext} X$ a compact subset, F a Fréchet space and f a bounded mapping in $\mathcal{C}_\alpha(K, F)$.

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Let X be a simplex, $K \subset \operatorname{ext} X$ a compact subset, F a Fréchet space and f a bounded mapping in $\mathcal{C}_\alpha(K, F)$. Then f can be extended to a mapping from $\mathfrak{A}_\alpha(X, \overline{\operatorname{co}} f(K))$.

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Theorem

Let K be a compact subset of a completely regular topological space Z, F be a Fréchet space and $f : K \to F$ be a bounded mapping in $\mathcal{C}_\alpha(K, F)$. Then there exists a mapping $h : Z \to F$ in $\mathcal{C}_\alpha(Z, F)$ extending f such that $h(Z) \subset \overline{\operatorname{co}} f(K)$.

Affine Jayne-Rogers selection result

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Example

There are simplices X_1, X_2 and upper semicontinuous mappings $\Gamma_i\colon X_i \to \mathbb{R}$ with closed values, bounded range and convex graph for $i = 1, 2$ such that the following assertions hold:

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(i) *X_1 is metrizable and Γ_1 admits no affine Baire-one selection.*
(ii) *X_2 is non-metrizable and Γ_2 admits no affine Borel selection.*

Thank you for your attention.