Boolean Topological Groups and Extremally Disconnected Groups

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All topological spaces are assumed to be completely regular and Hausdorff.

Definition

A topological space is said to be extremally disconnected if the closure of any open set in this space is open (or, equivalently, the closures of any two disjoint open sets are disjoint).

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Does there exist in ZFC *a nondiscrete extremally disconnected topological group?*

Malykhin: Any extremally disconnected topological group must contain an open Boolean subgroup.

Thus, the existence of an extremally disconnected topological group is equivalent to the existence of a Boolean extremally disconnected topological group.

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2. vector spaces over $\mathbb{Z}_2$;
3. free (algebraically).

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1969–1970: Sidney Morris introduced the notion of a variety of topological groups (this is a class of topological groups closed with respect to taking topological subgroups, topological quotient groups, and Cartesian products of groups with the product topology) and studied free objects of these varieties.

The free topological group $F(X)$ on a space X:

1. X is topologically embedded in $F(X)$ and
2. for any continuous map f of X to a topological group G, there exists a continuous homomorphism $\hat{f}\colon F(X) \to G$ for which $f = \hat{f} \restriction X$.

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The free Abelian topological group $A(X)$ on X:

any continuous map f of X to an *Abelian* topological group can be extended to a homomorphism.

The free Boolean topological group $B(X)$ on X:

any continuous map f of X to a *Boolean* topological group can be extended to a homomorphism.

Whenever X algebraically generates a group G, we can define the length of any $g \in G$ with respect to X:

1. the length of the identity element is set to 0;
2. the length of any nonidentity $g \in G$ with respect to X is the least (positive) integer n such that $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ for some $x_i \in X$ and $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots, n$.

We denote the set of elements of G of length at most k (with respect to X) by G_k for $k \in \omega$; then $G = \bigcup G_k$. Thus, we use $F_k(X)$, $A_k(X)$, and $B_k(X)$ to denote the sets of words of length at most k in $F(X)$, $A(X)$, and $B(X)$, respectively.

Comparison of Free, Free Abelian, and Free Boolean Topological Groups: Similarity

1. The sets $F_n(X)$, $A_n(X)$, and $B_n(X)$ are closed in the respective groups.
2. For any family $\{X_\alpha\colon \alpha \in A\}$ of spaces,
$$B\Big(\bigoplus_{\alpha\in A} X_\alpha\Big) \cong \sigma\square_{\alpha\in A} B(X_\alpha).$$
3. The free Boolean topological group of a nondiscrete space is never metrizable.
4. Let $Y \subset X$. The topological subgroup of $B(X)$ generated by Y is $B(Y)$ if and only if each bounded continuous pseudometric on Y can be extended to X.
5. If $\dim X = 0$, then $\operatorname{ind} B(X) = 0$.
6. Given a filter $\mathcal{F}$ on ω, $B(\omega_{\mathcal{F}})$ has the inductive limit topology if and only if $\mathcal{F}$ is a P-filter.

Comparison of Free, Free Abelian, and Free Boolean Topological Groups: Difference

1. The free Abelian topological group of any connected space has infinitely many connected components. The free Boolean topological group of any connected space has two connected components.
2. All finite powers X^n are contained in $F(X)$ and $A(X)$ as closed subspaces. Under CH, there exist an X such that X^2 is not contained in $B(X)$ as a subspace.
3. There exist spaces X and Y for which $B(X)$ and $B(Y)$ are toppologically isomorphic but $A(X)$ and $A(Y)$ (and $F(X)$ and $F(Y)$) are not.

Specifics of Boolean Topological Groups

There is a fundamental difference in the very topological-algebraic nature of free, free Abelian, and free Boolean groups:

1. Nontrivial free and free Abelian groups admit no compact group topologies. On the other hand, for any infinite cardinal κ, the direct sum $\bigoplus_{2^\kappa} \mathbb{Z}_2$ of 2^κ copies of $\mathbb{Z}_2$ (that is, the free Boolean group of rank 2^κ) is algebraically isomorphic to the Cartesian product $(\mathbb{Z}_2)^\kappa$ and, therefore, admits compact group topologies (e.g., the product topology).
2. The free and free Abelian groups are never finite, while the free Boolean group of any finite set is finite.

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2. The free and free Abelian groups are never finite, while the free Boolean group of any finite set is finite.
3. Any countable Boolean topological group has a closed discrete basis.
4. The study of extremally disconnected groups reduces to that of Boolean extremally disconnected topological groups thanks to Malykhin's theorem (that any extremally disconnected topological group contains an open Boolean subgroup).

Each free filter $\mathcal{F}$ on any set X is associated with $X_{\mathcal{F}} = X \cup \{*\}$ ($*$ is a point not belonging to X); all points of X are isolated and the neighborhoods of $*$ are $\{*\} \cup A$, $A \in \mathcal{F}$.

$B(X_{\mathcal{F}})$ is topologically isomorphic to the *Graev free topological group* $B_G(X_{\mathcal{F}})$ in which the only nonisolated point of $X_{\mathcal{F}}$ is the zero of $B(X_{\mathcal{F}})$.

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The point $*$, which is the zero element of $B(X_{\mathcal{F}})$, is identified with the empty set $\varnothing$, which belongs to $[X]^{<\omega}$ as the zero element.

Each $x \in X$ is identified with $\{x\} \in [X]^{<\omega}$.

We assume all filters $\mathcal{F}$ on ω to be free, i.e., to contain the Fréchet filter (of all cofinite sets).

A filter $\mathcal{F}$ on ω is said to be a P-filter if, for any family of $A_i \in \mathcal{F}$, $i \in \omega$, the filter $\mathcal{F}$ contains a *pseudointersection* of this family, i.e., a set $A \subset \omega$ such that $|A \setminus A_i| < \omega$ for all $i \in \omega$. For ultrafilters, this property is equivalent to being a P-point, or weakly selective, ultrafilter.

A filter $\mathcal{F}$ on ω is said to be Ramsey if for any family of $A_i \in \mathcal{F}$, $i \in \omega$, the filter $\mathcal{F}$ contains a *diagonal* of this family, i.e., a set $D \subset \omega$ such that, whenever $i, j \in D$ and $i < j$, we have $j \in A_i$. Ultrafilters with this property are known as Ramsey, or selective, ultrafilters.

We use the standard notation $[\omega]^{<\omega}$ for the set of all finite subsets of ω and $\omega^{<\omega}$ for the set of all finite sequences of elements of ω. Given $s, t \in [\omega]^{<\omega}$, $s \sqsubset t$ means that s is an initial segment of t, i.e., $s \subset t$ and all elements of $t \setminus s$ are greater than all elements of s. For $s \in [\omega]^{<\omega} \setminus \{\varnothing\}$ by $\max s$ we mean the greatest element of s in the ordering of ω. We also set $\max \varnothing = -1$.

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The Mathias forcing $\mathbb{M}(\mathcal{F})$ and (a modification of) the Laver forcing $\mathbb{L}(\mathcal{F})$ relative to a filter $\mathcal{F}$ determine two natural topologies on $[\omega]^{<\omega}$: the Mathias topology τ_M generated by the base

$$\{[s, A] \colon s \in [\omega]^{<\omega},\ A \in \mathcal{F}\},$$
$$\text{where} \quad [s, A] = \{t \in [\omega]^{<\omega} \colon s \sqsubset t,\ t \setminus s \subset A\},$$

and the Laver topology τ_L generated by all sets $U \subset [\omega]^{<\omega}$ such that

$$t \in U \implies \{n > \max t \colon t \cup \{n\} \in U\} \in \mathcal{F}.$$

The Mathias topology τ_M = the topology of the free Boolean linear topological group on $\omega_{\mathcal{F}}$ (linear groups are those with topology generated by subgroups): a base of neighborhoods of zero is formed by the sets $[\varnothing, A]$ with $A \in \mathcal{F}$, that is, by all subgroups generated by elements of $\mathcal{F}$.

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The Laver topology τ_L is the maximal invariant topology on $[\omega]^{<\omega}$ in which the filter $\mathcal{F}$ converges to zero. (An invariant topology is a topology with respect to which the group operation is separately continuous; groups with an invariant topology are said to be semitopological. The convergence of $\mathcal{F}$ to zero means that τ_L induces the initially given topology on $\omega_{\mathcal{F}}$.)

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The free group topology occupies an intermediate position between τ_M and τ_L.

Theorem (Thümmel, 2007)

For any filter on ω, the following conditions are equivalent:

1. $\mathcal{F}$ *is Ramsey;*
2. $\tau_M = \tau_{\mathrm{free}} = \tau_{\mathrm{indlim}} = \tau_L$;
3. τ_L *is a group topology;*
4. *for any sequence of* $A_i \in \mathcal{F}$, $i \in \omega$, *the set* $U = \{\varnothing\} \cup \bigcup_{i\in\omega}[i, A_i]$ *is open in* τ_{free}.

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Given a filter $\mathcal{F}$ *on* ω, *the group* $B(\omega_{\mathcal{F}})$ *is extremally disconnected if and only if* $\mathcal{F}$ *is a Ramsey ultrafilter.*

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Corollary (S.)

The free Boolean group on a nondiscrete countable space X *is extremally disconnected if and only if* X *is an almost discrete space associated with a Ramsey ultrafilter.*

Definition

An ultrafilter $\mathscr{U}$ on ω is

- a *P-point* if, for any partition $\{A_n : n \in \omega\}$ of ω such that $A_n \notin \mathscr{U}$ for any n, there exists an $A \in \mathscr{U}$ such that $|A \cap A_n| < \aleph_0$ for any n;
- Ramsey, or selective, if, for any partition $\{A_n : n \in \omega\}$ of ω such that $A_n \notin \mathscr{U}$ for any n, there exists an $A \in \mathscr{U}$ such that $|A \cap A_n| \leqslant 1$ for any n;

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- *Q*-point $=$ Ramsey $-$ *P*-point:
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- *Q-point* $=$ Ramsey $-$ *P*-point: for any partition $\{A_n : n \in \omega\}$ of ω such that A_n is finite for any n, there exists an $A \in \mathscr{U}$ such that $|A \cap A_n| \leqslant 1$ for any n;
- *rapid*, if, for any partition $\{A_n : n \in \omega\}$ of ω such that A_n is finite for any n, there exists an $A \in \mathscr{U}$ such that $|A \cap A_n| \leqslant n$ for any n $\iff$ every function $\omega \to \omega$ is majorized by the increasing enumeration of some element of $\mathcal{U}$

CH $\implies$ $\exists$ selective ultrafilters, $P \neq Q \neq \text{selective} \neq P$

ZFC $\implies$ $\exists$ an ultrafilter which is neither a P-point nor a Q-point

Shelah: There is a model in which $\nexists$ P-point ultrafilters

Miller: In Laver's model $\nexists$ Q-points (but $\exists$ P-points)

Old problem: Does there exist a model in which there are no P-points and no Q-points?

Theorem (Reznichenko + S., July 2016)

The existence of a countable nondiscrete extremally disconnected group G implies the existence of either a rapid ultrafilter or a P-point ultrafilter.

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Let G be a topological group with identity element e for which the filter of neighborhoods of e is not rapid. Suppose that $S \subset G$, $e \in \overline{S} \setminus S$, and $\{W_n \subset G : n \in \omega\}$ is a sequence of sets such that $W_n \cap W_n W_n^{-1} = \varnothing$ for all $n \in \omega$. Then there exists a sequence $\xi = \{x_n \in SS^{-1} : n \in \omega\}$ such that $e \in \overline{\xi}$ and $|\xi \cap W_n| < \omega$ for all $n \in \omega$.

Theorem (Reznichenko + S., July 2016)

Suppose that G is a countable extremally disconnected topological group with zero 0 for which the filter of neighborhoods of 0 is not rapid and $(U_n)_n$ is a decreasing sequence of clopen neighborhoods of 0 for which $U_{n+1} + U_{n+1} \subset U_n$ and $\bigcap_n U_n = \{e\}$. Let $C_n = U_n \setminus U_{n+1}$. Then

$$p = \{\{n : U \cap C_n \neq \varnothing\} : U \text{ is a neighborhood of } e\}$$

is a P-point ultrafilter.

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If there exist no rapid ultrafilters and G is a countable Boolean extremally disconnected group, then

1. G contains no open subgroups;

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If there are no rapid ultrafilters, then any countable topological group contains a nonclosed discrete subset with only one limit point.

If there exist no rapid ultrafilters and G is a countable Boolean extremally disconnected group, then

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If there exist no rapid ultrafilters and G is a countable Boolean extremally disconnected group, then

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1. G contains no open subgroups;
2. any linearly independent subset of G is closed and discrete;
3. the intersection of finitely many nondiscrete subgroups of G is a nondiscrete subgroup;
4. if $A, B \subset G$ and $\overline{A} \cap \overline{B} = \varnothing$, then $(A + A) \cap (B + B) = \varnothing$.

THANK YOU