Minimality of the semidirect product

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Definition

Let G be a Hausdorff topological group.

1. *G is* **minimal** *if it does not admit a strictly coarser Hausdorff group topology (Stephenson, Doïtchinov).*
2. *A subgroup $H \leq G$ is* **essential** *in G if $H \cap N \neq \{e\}$ for every closed nontrivial normal subgroup N of G.*

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Minimality Criterion (Banaschewski, Stephenson, Prodanov)

Let G be a topological group and H its dense subgroup. Then H is minimal if and only if G is minimal and H is essential in G.

Minimal precompact groups

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2. $(\mathbb{Z}, \tau_p)$ (Prodanov)

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Theorem (Prodanov, Stoyanov, 1984)

Every minimal abelian group is precompact.

Nonabelian minimal groups

1. $\mathbb{R} \leftthreetimes \mathbb{R}_+$ (Dierolf, Schwanengel, 1979).
2. Every connected semi-simple Lie group with finite center, e.g. $SL_n(\mathbb{R})$ where $n > 1$ (Remus, Stoyanov, 1991).
3. The pointwise topology is the **minimum** Hausdorff group topology on $S(X)$ (Gaughan, 1967).
4. Extension of (3) to every subgroup of $S(X)$ containing the permutations of finite support (Banakh, Guran, Protasov, 2012).

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Minimality of $\mathrm{H}(X)$

1. $\mathrm{H}([0,1]^n)$ is minimal iff $n = 1$ (Gamarnik, 1991).
2. For $\mathrm{H}[0,1]$ and $\mathrm{H}(\mathbb{S}^1)$, τ_{co} is the minimum Hausdorff group topology (Gartside, Glyn, 2003). This result was extended to some compact connected LOTS (Megrelishvili, Polev, 2015).
3. $\mathrm{H}(2^\omega)$ is minimal (Gamarnik, 1991). More generally, $\mathrm{H}(X)$ is minimal for every h-homogeneous compact space X (Uspenskij, 2001).
4. Let X be the n-dimensional Menger universal continuum ($n > 0$). Then $\mathrm{H}(X)$ is not minimal (van Mill, 2010).
5. For every compact metrizable space X containing an open n-cell, $n \geq 2$, $\mathrm{H}(X)$ has no minimum Hausdorff group topology. For every compact metrizable space X containing a dense open one-manifold, $\mathrm{H}(X)$ has the minimum topology (Chang, Gartside, 2015).

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Semidirect products

1. $G \leftthreetimes P$ can be minimal even if G and P are not minimal. For example, $\mathbb{R} \leftthreetimes \mathbb{R}_+$ is minimal.
2. There exists a precompact minimal group G and a two element subgroup $P \leq \mathrm{Aut}(G)$ such that $G \leftthreetimes P$ is not minimal.

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Example (Eberhardt, Dierolf, Schwanengel, 1980)

- I is an infinite index set, A_5^I is equipped with the product topology.
- $G = \{(x_i)_{i \in I} : x_i \in A_5 \wedge |i : x_i \neq e| < \infty\}$ is a minimal precompact group being essential dense subgroup of the compact group A_5^I.
- Now choose an element $z \in A_5$ of order two. Let $P := \{Id, \gamma_z\} \leq \mathrm{Aut}(G)$, where γ_z is the inner automorphism defined by z.
- Clearly, $G \leftthreetimes P$ is a dense subgroup of the compact group $A_5^I \leftthreetimes P$. $G \leftthreetimes P$ is not essential in $A_5^I \leftthreetimes P$.
- Indeed, the 2-element group generated by $((z)_{i \in I}, \gamma_z)$ is a closed normal subgroup of $A_5^I \leftthreetimes P$ that intersects $G \leftthreetimes P$ trivially. By the minimality criterion $G \leftthreetimes P$ is not minimal.

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This example shows that the group $G \leftthreetimes P$ may fail to be minimal, even if G and P are minimal. However, adding the requirement of completeness of G, one has the following:

Theorem (Eberhardt, Dierolf, Schwanengel, 1980)

If G is complete (with respect to its two-sided uniformity), then $G \leftthreetimes P$ is minimal for minimal groups G and P.

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A natural question arises:

Question

Is there a compact group G and a closed subgroup $P \leq \mathrm{Aut}(G)$ such that $G \leftthreetimes P$ is not minimal?

Remark

It is important to note that there are compact groups G such that $\mathrm{Aut}(G)$ is not minimal. Indeed, one may take $G = (\mathbb{Q}, \text{discrete})^*$, that is the Pontryagin dual of the discrete group $\mathbb{Q}$ (Dikranjan, Megrelishvili).

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Theorem

If G is a compact topological group, then $G \leftthreetimes P$ is minimal for every closed subgroup P of $\mathrm{Aut}(G)$.

Using this theorem and the minimality criterion we obtain the following:

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Abelian case

Theorem

If G is a compact abelian topological group, then $G \leftthreetimes P$ is minimal for every (not necessarily closed) subgroup P of $\mathrm{Aut}(G)$.

Remark

In fact, we prove a bit more. Let G be a compact (not necessarily abelian) topological group, and $P \leq \mathrm{Aut}(G)$ such that $\overline{P}$ does not contain a nontrivial inner automorphism. Then we can show that the dense subgroup $G \leftthreetimes P$ is essential in the minimal group $G \leftthreetimes \overline{P}$.

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Definition (Megrelishvili)

1. *Let $\pi : G \times X \to X$ be an action of a topological group (G, τ) on a Hausdorff uniform space $(X, \mathfrak{U})$. The uniformity (or, the action) is π-**uniform** if*

$$\forall g_0 \in G \ \forall \varepsilon \in \mathfrak{U} \ \exists \delta \in \mathfrak{U}, \ \exists O \in N_{g_0}(\tau)$$

$$(x, y) \in \delta, g \in O \Rightarrow (gx, gy) \in \varepsilon$$

2. *Let X be a compact space and G a subgroup of $\mathrm{H}(X)$. A Hausdorff group topology τ on G is said to be π-**uniform** if the natural action $(G, \tau) \times X \to X$ is π-uniform.*

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- The notion of a π-uniform action was originally used to study compactifications of G-spaces.
- Later it was employed by Gamarnik to prove that for a compact space X, the compact-open topology on $\mathrm{H}(X)$ is minimal within the class of π-uniform topologies. We extend this result to every closed subgroup of $\mathrm{H}(X)$.

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Theorem

Let (X, τ) be a compact topological space and let P be a closed subgroup of $\mathrm{H}(X)$, the group of all homeomorphisms of X. Then the compact-open topology τ_{co} is minimal within the class of π-uniform topologies on P.

As a corollary we get the following:

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If K is a compact topological group and P is a closed subgroup of $\mathrm{Aut}(K)$, then the compact-open topology is minimal within the class of π-uniform topologies on P.

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If K is a compact topological group and P is a closed subgroup of $\mathrm{Aut}(K)$, then the compact-open topology is minimal within the class of π-uniform topologies on P.

For a topological group (G, γ) and its subgroup H denote by γ/H the natural quotient topology on the coset space G/H.

Merson's Lemma

Let (G, γ) be a (not necessarily Hausdorff) topological group and H be a (not necessarily closed) subgroup of G. If $\gamma_1 \subseteq \gamma$ is a coarser group topology on G such that $\gamma_1|_H = \gamma|_H$ and $\gamma_1/H = \gamma/H$, then $\gamma_1 = \gamma$.

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Main result

Theorem

If G is a compact topological group, then $G \leftthreetimes P$ is minimal for every closed subgroup P of $\mathrm{Aut}(G)$.

Sketch

The structure of the proof

- γ is the product topology on $G \leftthreetimes P$.
- Assume that $\gamma_1 \subseteq \gamma$ is a coarser Hausdorff group topology on $G \leftthreetimes P$. Clearly, $\gamma_1|_G = \gamma|_G$. We want to show that $\gamma_1/G = \gamma/G$ and conclude the proof using Merson's Lemma.

$\gamma_1/G = \gamma/G = \tau_{co}$

- The action

$$\alpha : (P, \gamma_1/G) \times (G, \gamma_1|_G) \to (G, \gamma_1|_G)$$

is α-uniform and γ_1/G is an α-uniform topology on P.
- Since $\gamma_1/G \subseteq \gamma/G = \tau_{co}$ and τ_{co} is minimal within the class of α-uniform topologies on P, we have $\gamma_1/G = \gamma/G$.

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The question which remains open is:

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