Setwise and Pointwise Betweenness via Hyperspaces

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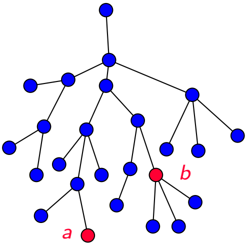
Overview

1. Motivation
2. Setwise Betweenness via 2^X and $\mathcal{F}_n(X)$
3. Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Motivation

An intuitive view of betweenness arises naturally in any order-theoretic structure; given a preorder $\leq$ on a set X, with $a, b, c \in X$ such that $a \leq c \leq b$, we say that "c is between a and b".

- Let $(X, \leq)$ be a partially ordered set. Define for $a \leq b$, $[a, b]_O = \{c \in X : a \leq c \leq b\}$.
  If $(X, \leq)$ is a tree with a common lower bound d of a, b.
  $O(a, b, d) = [d, a]_O \bigcup [d, b]_O$.



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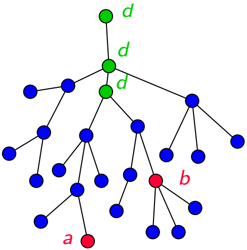
  Define $[a, b]_T = \{c \in O(a, b, d) : d \leq a, b\}$



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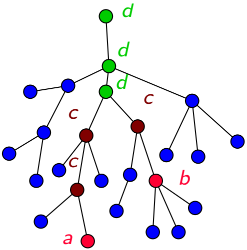
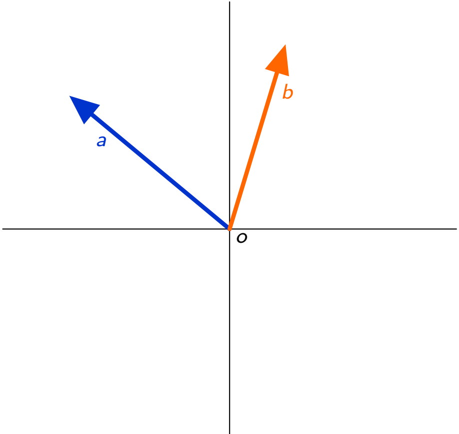


Motivation

- Let X be a vector space over the real field $\mathbb{R}$ and let $a, b \in X$. The convex interval can be defined as follows: A vector $c \in X$ is between a and b if $c \in [a, b]_{conv} = \{at + (1 - t)b : t \in [0, 1]\}$. So $[a, b]_{conv}$ is the set of all convex combinations of a and b.



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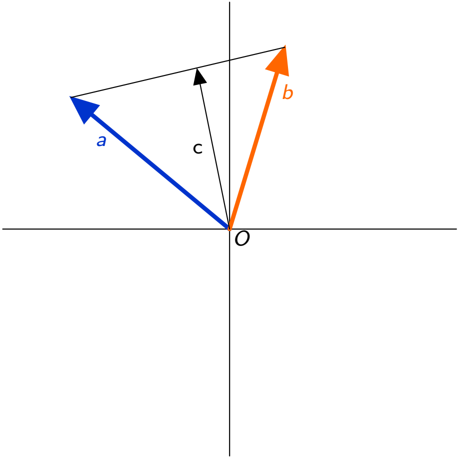

Road Systems and Pointwise Betweenness

Paul Bankston introduced the following definitions:

Definition

A road system is a pair $\langle X, \mathcal{R} \rangle$, where X is a nonempty set and $\mathcal{R}$ is a collection of nonempty subsets of X -called the roads- such that:

1. For each $a \in X$, the singleton set $\{a\}$ is a road.
2. Each two points $a, b \in X$ belong to at least one road.

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Definition

Let $\langle X, \mathcal{R} \rangle$ be a road system and $a, b, c \in X$. Then $c \in [a, b]_{\mathcal{R}}$ if every road containing a and b also contain c. Then

$$c \in [a, b]_{\mathcal{R}} \text{ if } c \in \bigcap \{R \in \mathcal{R} : R \in \mathcal{R}(a, b)\}$$

where $\mathcal{R}(a, b)$ denotes the set of roads that contain both a and b

Road Systems and Setwise Betweenness

There is a natural generalisation from pointwise betweenness to setwise betweenness as follows:

Definition

Let $\langle X, \mathcal{R} \rangle$ be a road system with $a, b \in X$ and $\emptyset \neq C \subseteq X$. We say that C is between a and b if $C \bigcap R \neq \emptyset$ for all $R \in \mathcal{R}(a, b)$

Vietoris Hyperspace (2^X)

Definition

Let X be a T_1 space. The Vietoris topology 2^X on $CL(X)$, the collection of all non-empty closed subsets of X, is the one generated by sets of the form

$$U^+ = \{A \in CL(X) : A \subset U\}$$
$$U^- = \{A \in CL(X) : A \bigcap U \neq \emptyset\}$$

where U is an open subset of X.
A basis of the Vietoris topology consists of the collection of sets of the form
$\langle U_1, U_2, ..., U_n\rangle = \{A \in CL(X) : A \subseteq \bigcup_{i=1}^{n} U_i \text{ and if } i \leq n, A \bigcap U_i \neq \emptyset\}$
where $U_1, U_2, ..., U_n$ are non-empty open subsets of X.

n-fold Symmetric Product hyperspace $\mathcal{F}_n(X)$

Definition

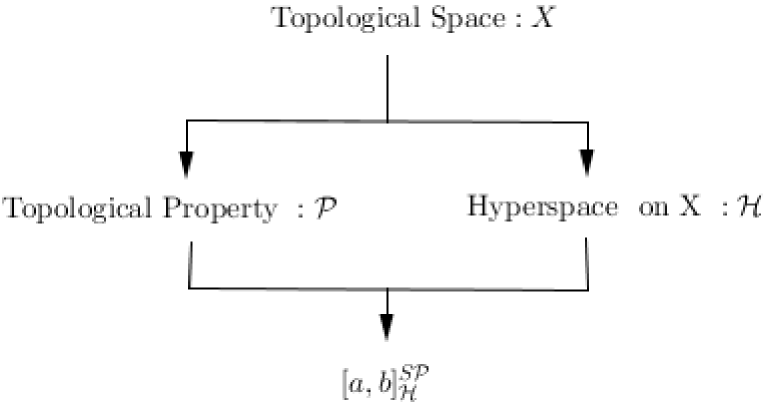
Let X be a T_1 space, the hyperspace $\mathcal{F}_n(X)$, called n-fold symmetric product of X, is a subspace of the Vietoris space 2^X defined as follows

$$\mathcal{F}_n(X) = \{A \in X : |A| \leq n\}$$

Some properties of $\mathcal{F}_n(X)$

- $\mathcal{F}_1(X) \cong X$
- $\mathcal{F}_n(X) \subseteq \mathcal{F}_{n+1}(X)$
- If X is a Hausdorff space then $\mathcal{F}_n(X)$ is a closed subspace in the Vietoris hyperspace.

Setwise Betweenness via 2^X and $\mathcal{F}_n(X)$



Setwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Notation

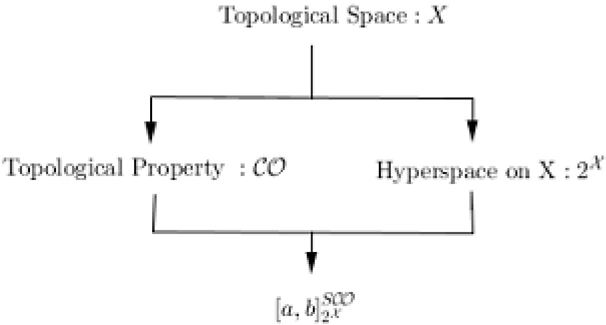
Let X be a topological space and $a, b \in X$, the collection of sets that satisfies a topological property $\mathcal{P}$ forms a road system.
The collection of sets that contain a and b and satisfy a topological property $\mathcal{P}$ is denoted by $\mathcal{P}(a, b)$.

Definition

Let X be a T_1 space. Define the setwise interval with respect to a property $\mathcal{P}$ and a hyperspace $\mathcal{H}$ as follows:

$$[a, b]_{\mathcal{H}}^{S\mathcal{P}} = \{C \in \mathcal{H} : C \bigcap K \neq \emptyset \text{ for every } K \in \mathcal{P}(a, b)\}$$

The Setwise Interval $[a, b]_{2^X}^{SCO}$



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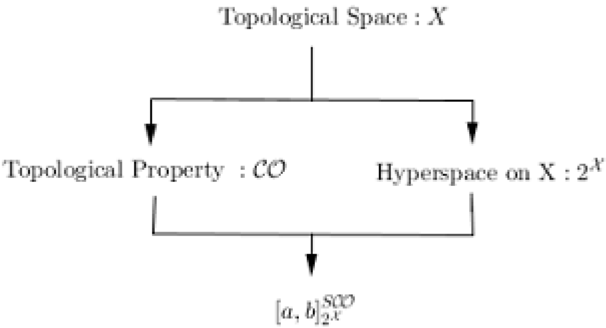


Definition

Let X be a topological space. Define the setwise interval with respect to the Vietoris hyperspace 2^X as follows:

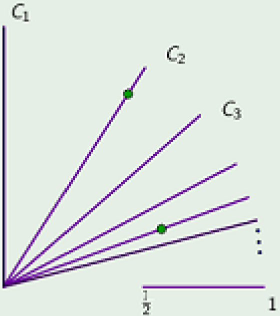
$$[a, b]_{2^X}^{\mathcal{SCO}} = \{C \in 2^X : C \bigcap K \neq \emptyset \text{ for every } K \in \mathcal{CO}(a, b)\}$$

where $\mathcal{CO}(a, b)$ the collection of all connected sets that contains a and b.

The Setwise Interval $[a, b]^{SCO}_{2^X}$

Example

Let $X = C \bigcup B$ be a subspace of the $\mathbb{R}^2$ where $C = [\frac{1}{2}, 1]$ and $B = \{(0,0)\} \bigcup_{n=1}^{\infty} C_n$. Now if $a \in C_i$ and $b \in C_j$ with $i \neq j$ then for a $A \in 2^X$ to be lie in the interval $[a, b]^{SCO}_{2^X}$ it is necessary and sufficient that $(0,0) \in A$.



Some Properties of The Interval $[a, b]_{2^X}^{\mathcal{SCO}}$

Let X be a T_1 space with $a, b \in X$. Then

1. $\{a\}, \{b\} \in [a, b]_{2^X}^{\mathcal{SCO}}$
2. $[a, b]_{2^X}^{\mathcal{SCO}} \subseteq [a, a]_{2^X}^{\mathcal{SCO}}, [b, b]_{2^X}^{\mathcal{SCO}}$

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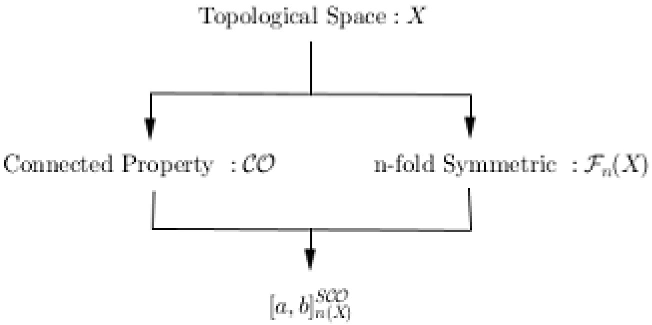
Theorem

If $f : X \longrightarrow Y$ be a homeomorphism then

$$f([a,b]_{2^X}^{\mathcal{SCO}}) = [f(a), f(b)]_{2^Y}^{\mathcal{SCO}}$$

The Setwise Interval $[a, b]^{SCO}_{n(X)}$



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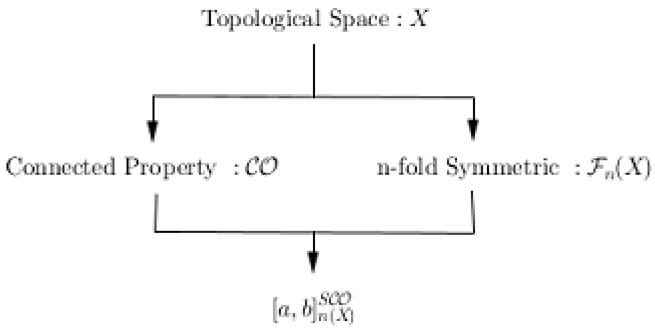


Definition

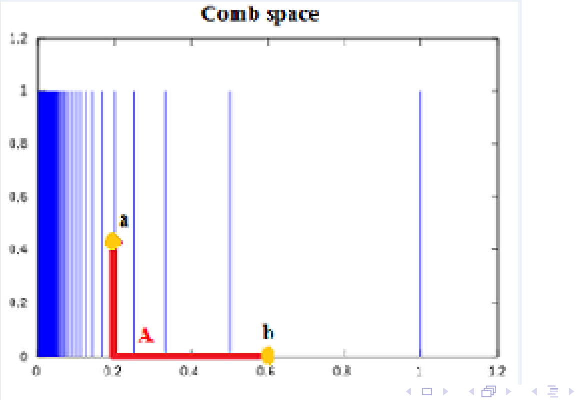
Let X be a topological space. Define the setwise interval with respect to the n-fold symmetric product hyperspace $\mathcal{F}_n(X)$ as follows:

$$[a, b]^{SCO}_{n(X)} = \{C \in \mathcal{F}_n(X) : C \bigcap K \neq \emptyset \text{ for every } K \in \mathcal{CO}(a, b)\}$$

The Setwise Interval $[a,b]^{SCO}_{n(X)}$

Example

Let X be the comb space and $A = \{[x, 0] \bigcup [0.2, y] : \text{ where } 0.2 \leq x \leq 0.6 \text{ and } 0 \leq y \leq 0.4\}$. It is clear that $A \in \mathcal{CO}(a, b)$. Thus for $C \in \mathcal{F}_n(X)$ to lie between a and b, i.e. to be sure that $C \in [a, b]^{P\mathcal{CO}}_{n(X)}$ it is enough for C to intersect A.



Some Properties of The Interval $[a, b]^{\mathcal{SCO}}_{n(X)}$ continue ....

Some properties of the setwise interval $[a, b]^{\mathcal{SCO}}_{n(X)}$

Let X be a topological space with $a, b \in X$. Then

1. $\{a\}, \{b\} \in [a, b]^{\mathcal{SCO}}_{n(X)}$
2. $[a, b]^{\mathcal{SCO}}_{n(X)} \subseteq [a, a]^{\mathcal{SCO}}_{n(X)}, [b, b]^{\mathcal{SCO}}_{n(X)}$
3. For $n \geq 3$, we have $[a, b]^{\mathcal{SCO}}_{n(X)} \bigcap [b, c]^{\mathcal{SCO}}_{n(X)} \neq \emptyset$
4. $[a, b]^{\mathcal{SCO}}_{1(X)} \subseteq [a, b]^{\mathcal{SCO}}_{2(X)} \subseteq ... \subseteq [a, b]^{\mathcal{SCO}}_{n(X)}$

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4. $[a, b]^{SCO}_{1(X)} \subseteq [a, b]^{SCO}_{2(X)} \subseteq \ldots \subseteq [a, b]^{SCO}_{n(X)}$

Proposition

Proposition: Let X be a topological space with $a, b \in X$ and $C_i \in \mathcal{F}_n(X)$ for $i = 1, 2, \ldots$ such that $C_1 \subset C_2 \subset \ldots.$ If $C_1 \in [a, b]^{SCO}_{n(X)}$ then $C_i \in [a, b]^{SCO}_{n(X)}$ for each $i = 2, 3, \ldots$

Some Properties of The Interval $[a, b]^{\mathcal{SCO}}_{n(X)}$ continue ....

Theorem

If $f : X \longrightarrow Y$ *be a homeomorphism then*

$$f([a, b]^{\mathcal{SCO}}_{n(X)}) = [f(a), f(b)]^{\mathcal{SCO}}_{n(Y)}$$

Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Definition

Let X be a topological space with $x \in X$. The hyperstar collection of x with respect to a hyperspace $\mathcal{H}$ is defined by

$$st(x, \mathcal{H}) = \{C \in \mathcal{H} : x \in C\}$$

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- $st(x, 2^X) = \{C \in 2^X : x \in C\}$
- $st(x, \mathcal{F}_n(X)) = \{C \in \mathcal{F}_n(X) : x \in C\}$

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- $st(x, \mathcal{F}_n(X)) = \{C \in \mathcal{F}_n(X) : x \in C\}$

Some properties of $st(x, \mathcal{F}_n(X))$

- $st(x, \mathcal{F}_1(X)) = \{\{x\}\}$
- $st(x, \mathcal{F}_n(X)) \subset st(x, \mathcal{F}_{n+1}(X))$

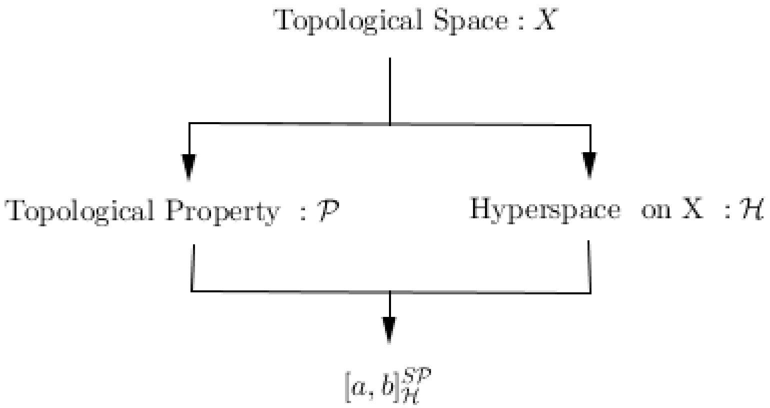
Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Definition

Let X be a T_1 space. We define the hyperstar collection of a set $C \subset X$ as follows:

$$st(C, \mathcal{F}_n(X)) = \bigcup_{c \in C} st(c, \mathcal{F}_n(X))$$

Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$



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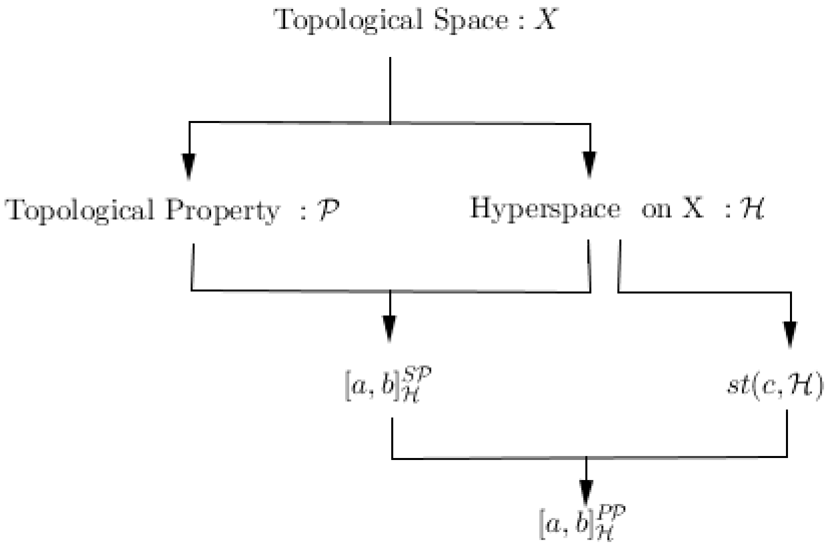

Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Definition

Let X be a topological space with $a, b, c \in X$. We say that c lies between a and b with respect to a hyperspace $\mathcal{H}$ (denoted by $c \in [a, b]_{\mathcal{H}}^{\mathcal{PP}}$) if $st(c, \mathcal{H}) \subset [a, b]_{\mathcal{H}}^{\mathcal{SP}}$.

Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

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Pointwise interval via 2^X

Definition

Let X be a topological space. Define the pointwise interval with respect to 2^X as follows:

$$[a, b]_{2^X}^{\mathcal{PCO}} = \{c \in X : st(c, 2^X) \subset [a, b]_{2^X}^{\mathcal{SCO}}\}$$

Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Pointwise interval via $\mathcal{F}_n(X)$

Definition

Let X be a topological space. Define the pointwise interval with respect to $\mathcal{F}_n(X)$ as follows:

$$[a,b]^{PCO}_{n(X)} = \{c \in X : st(c, \mathcal{F}_n(X)) \subset [a,b]^{SCO}_{n(X)}\}$$

Pointwise Betweenness via 2^X and $\mathcal{F}_n(X)$

Pointwise interval via $\mathcal{F}_n(X)$

Definition

Let X be a topological space. Define the pointwise interval with respect to $\mathcal{F}_n(X)$ as follows:

$$[a,b]^{PCO}_{n(X)} = \{c \in X : st(c, \mathcal{F}_n(X)) \subset [a,b]^{SCO}_{n(X)}\}$$

Some properties

- $\{a,b\} \subset [a,b]^{PCO}_{n(X)}$
- $[a,b]^{PCO}_{n(X)} = [b,a]^{PCO}_{n(X)}$
- $[a,b]^{PCO}_{n(X)} \subset [a,b]^{PCO}_{n+1(X)}$
- Let $f : X \to Y$ be a homeomorphism map, then

$$f([a,b]^{PCO}_{n(X)}) = [f(a), f(b)]^{PCO}_{n(Y)}$$

A New Set Arose from Betweenness Setwise Interval $[a, b]^{\mathcal{SCO}}_{n(X)}$

Definition

Let X be a topological space with $a, b, c \in X$.

$$C^{n(X)}_{a,b} = \{c \in X : c \in [a, b]^{\mathcal{PCO}}_{\mathcal{F}_n(X)}\}$$

A New Set Arose from Betweenness Setwise Interval $[a,b]^{\mathcal{SCO}}_{n(X)}$

Definition

Let X be a topological space with $a, b, c \in X$.

$$C^{n(X)}_{a,b} = \{c \in X : c \in [a,b]^{\mathcal{PCO}}_{\mathcal{F}_n(X)}\}$$

Let X be a topological space with $a, b \in X$. Then

1. $a, b \in C^{n(X)}_{a,b}$
2. If $|C^{n(X)}_{a,b}| \leq n$ then $C^{n(X)}_{a,b} \in [a,b]^{\mathcal{SCO}}_{n(X)}$
3. $C^{n(X)}_{a,b} \subseteq C^{n+1(X)}_{a,b}$
4. $C^{n(X)}_{a,b} \subset C^{n(X)}_{a,a}, C^{n(X)}_{b,b}$

A New Set Arose from Betweenness Setwise Interval $[a, b]^{\mathcal{SCO}}_{n(X)}$

Definition

Let X be a topological space with $a, b, c \in X$. We define the following set:

$$C^{n(X)}_{a,b} = \{c \in X : c \in [a, b]^{\mathcal{PCO}}_{n(X)}\}$$

Let X be a topological space with $a, b \in X$. Then

1. $a, b \in C^{n(X)}_{a,b}$
2. If $|C^{n(X)}_{a,b}| \leq n$ then $C^{n(X)}_{a,b} \in [a, b]^{\mathcal{SCO}}_{n(X)}$
3. $C^{n(X)}_{a,b} \subseteq C^{n+1(X)}_{a,b}$
4. $C^{n(X)}_{a,b} \subset C^{n(X)}_{a,a}, C^{n(X)}_{b,b}$

Theorem

If $f : X \longrightarrow Y$ be a homeomorphism between two topological spaces then

$$f(C^{n(X)}_{a,b}) = C^{n(Y)}_{f(a),f(b)}$$

Thank You