Selectively sequentially pseudocompact group topologies on abelian groups

Alejandro Dorantes-Aldama, Dmitri Shakhmatov

TOPOSYM 2016

Let us recall two well-known compactness-type properties.

Definition

A topological space X is called:

(i) *sequentially compact* if every sequence in X has a convergent subsequence;

(ii) *pseudocompact* if every continuous real-valued function defined on X is bounded.

The following property is perhaps less familiar.

Definition (Artico, Marconi, Pelant, Rotter, Tkachenko; Dow, Porter, Stephenson, Woods)

A space X is called *sequentially pseudocompact* if for every (pairwise disjoint) family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood W of x.

Let us recall two well-known compactness-type properties.

Definition

A topological space X is called:

(i) *sequentially compact* if every sequence in X has a convergent subsequence;

(ii) *pseudocompact* if every continuous real-valued function defined on X is bounded.

The following property is perhaps less familiar.

Definition (Artico, Marconi, Pelant, Rotter, Tkachenko; Dow, Porter, Stephenson, Woods)

A space X is called *sequentially pseudocompact* if for every (pairwise disjoint) family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood W of x.

Proposition (Lipparini, 2012)

For every space X, the following conditions are equivalent:

(i) *X is sequentially pseudocompact;*

(ii) *for every sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood W of x.*

Two "selective" pseudocompactness-type properties

The (equivalent) properties from the next theorem can be considered "selective" properties, as they all involve a selection of a point from each set of a countable sequence of non-empty open sets in such a way that the resulting sequence satisfies some condition agreed in advance.

Proposition (Lipparini, 2012)

For every space X, the following conditions are equivalent:

(i) *X is sequentially pseudocompact;*

(ii) *for every sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood W of x.*

Two "selective" pseudocompactness-type properties

The (equivalent) properties from the next theorem can be considered "selective" properties, as they all involve a selection of a point from each set of a countable sequence of non-empty open sets in such a way that the resulting sequence satisfies some condition agreed in advance.

Theorem

For every space X, the following conditions are equivalent:

(i) *for each sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X, one can chose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ such that the set $\{x_n : n \in \mathbb{N}\}$ is not closed in X;*

(ii) *for each sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X, one can choose a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that $\mathrm{cl}_X(\{x_n : n \in \mathbb{N}\}) \setminus \bigcup_{n\in\mathbb{N}} U_n \neq \emptyset$;*

(iii) *for each sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can chose a point $x \in X$ and a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : x_n \in W\}$ is infinite for every open neighborhood W of x;*

(iv) *for each sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X on can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$, select a free ultrafilter p on $\mathbb{N}$ and find a point $x \in X$ such that $\{n \in \mathbb{N} : x_n \in W\} \in p$ for every open neighborhood W of x.*

Definition

A space X satisfying any (and then all) of the equivalent conditions in the above theorem will be called *selectively pseudocompact*.

The property from item (iv) of the above theorem appeared recently in the papers of García-Ferreira, Ortiz-Castillo and García-Ferreira, Tomita under the name "strong pseudocompactness". The property from item (ii) of the theorem above appeared under the same name in the abstract of the paper by García-Ferreira, Tomita. Since the term "strongly pseudocompact" is used by Arkhangel'skii, Genedi and Dikranjan to denote two different properties, we propose a new name for this property reflecting also its "selective" nature.

Definition

A space X satisfying any (and then all) of the equivalent conditions in the above theorem will be called *selectively pseudocompact*.

The property from item (iv) of the above theorem appeared recently in the papers of García-Ferreira, Ortiz-Castillo and García-Ferreira, Tomita under the name "strong pseudocompactness". The property from item (ii) of the theorem above appeared under the same name in the abstract of the paper by García-Ferreira, Tomita. Since the term "strongly pseudocompact" is used by Arkhangel'skii, Genedi and Dikranjan to denote two different properties, we propose a new name for this property reflecting also its "selective" nature.

Definition

A space X satisfying any (and then all) of the equivalent conditions in the above theorem will be called *selectively pseudocompact*.

The property from item (iv) of the above theorem appeared recently in the papers of García-Ferreira, Ortiz-Castillo and García-Ferreira, Tomita under the name "strong pseudocompactness". The property from item (ii) of the theorem above appeared under the same name in the abstract of the paper by García-Ferreira, Tomita. Since the term "strongly pseudocompact" is used by Arkhangel'skii, Genedi and Dikranjan to denote two different properties, we propose a new name for this property reflecting also its "selective" nature.

Definition

A space X satisfying any (and then all) of the equivalent conditions in the above theorem will be called *selectively pseudocompact*.

The property from item (iv) of the above theorem appeared recently in the papers of García-Ferreira, Ortiz-Castillo and García-Ferreira, Tomita under the name "strong pseudocompactness". The property from item (ii) of the theorem above appeared under the same name in the abstract of the paper by García-Ferreira, Tomita. Since the term "strongly pseudocompact" is used by Arkhangel'skii, Genedi and Dikranjan to denote two different properties, we propose a new name for this property reflecting also its "selective" nature.

To the best of our knowledge, the following "selective" property is new.

Definition

We shall call a space X *selectively sequentially pseudocompact* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence.

Proposition

For every topological space X, the following conditions are equivalent:

(i) *X is selectively sequentially pseudocompact;*

(ii) *for every sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X, one can choose $x_n \in U_n$ for all $n \in \mathbb{N}$ such that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence.*

To the best of our knowledge, the following "selective" property is new.

Definition

We shall call a space X *selectively sequentially pseudocompact* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence.

Proposition

For every topological space X, the following conditions are equivalent:

(i) *X is selectively sequentially pseudocompact;*

(ii) *for every sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X, one can choose $x_n \in U_n$ for all $n \in \mathbb{N}$ such that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence.*

For every sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ such that:

- the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence (selectively sequentially pseudocompact);
- the set $\{x_n : n \in \mathbb{N}\}$ is not closed (selectively pseudocompact).

sequentially compact $\longrightarrow$ countably compact

$\downarrow$ $\qquad\qquad\qquad$ $\downarrow$

selectively sequentially pseudocompact $\longrightarrow$ selectively pseudocompact

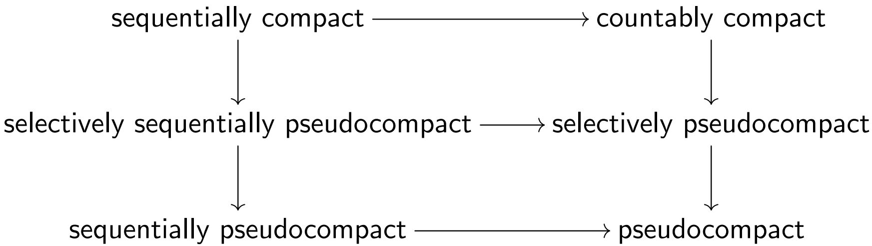
$\downarrow$ $\qquad\qquad\qquad$ $\downarrow$

sequentially pseudocompact $\longrightarrow$ pseudocompact

Can everyone see now why the name “selectively pseudocompact” is preferable to “strongly pseudocompact”?

For every sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ such that:

- the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence (selectively sequentially pseudocompact);
- the set $\{x_n : n \in \mathbb{N}\}$ is not closed (selectively pseudocompact).



Can everyone see now why the name "selectively pseudocompact" is preferable to "strongly pseudocompact"?

For every sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ such that:

- the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence (selectively sequentially pseudocompact);
- the set $\{x_n : n \in \mathbb{N}\}$ is not closed (selectively pseudocompact).

$$\begin{array}{ccc}
\text{sequentially compact} & \longrightarrow & \text{countably compact} \\
\downarrow & & \downarrow \\
\text{selectively sequentially pseudocompact} & \longrightarrow & \text{selectively pseudocompact} \\
\downarrow & & \downarrow \\
\text{sequentially pseudocompact} & \longrightarrow & \text{pseudocompact}
\end{array}$$

Can everyone see now why the name "selectively pseudocompact" is preferable to "strongly pseudocompact"?

- There is a pseudocompact space X such that all its countable subsets are closed and C^*-embedded (Shakhmatov). This X is (pseudocompact but) not selectively pseudocompact.
- Examples of selectively pseudocompact spaces that are not countably compact were constructed by García-Ferreira, Ortiz-Castillo.
- Dow, Porter, Stephenson, Woods showed that the Stone-Čech compactification $\beta\omega$ of ω is not sequentially pseudocompact. Hence $\beta\omega$ is a selectively pseudocompact space which is not sequentially pseudocompact.

$$\begin{array}{ccc}
\text{sequentially compact} & \longrightarrow & \text{countably compact} \\
\downarrow & & \downarrow \\
\text{selectively sequentially pseudocompact} & \longrightarrow & \text{selectively pseudocompact} \\
\downarrow & & \downarrow \\
\text{sequentially pseudocompact} & \longrightarrow & \text{pseudocompact}
\end{array}$$

Remark

Every infinite selectively sequentially pseudocompact space has a non-trivial convergent sequence.

- Let G be a pseudocompact group without non-trivial convergent sequences. (First example of such a group was given by Sirota.) Then G is sequentially pseudocompact (by a theorem of Artico, Marconi, Pelant, Rotter, Tkachenko cited below), yet it is not selectively sequentially pseudocompact.
- There exists a countably compact space which is not selectively sequentially pseudocompact (take infinite countaby compact subset of $\beta\omega$).

- Let G be a pseudocompact group without non-trivial convergent sequences. (First example of such a group was given by Sirota.) Then G is sequentially pseudocompact (by a theorem of Artico, Marconi, Pelant, Rotter, Tkachenko cited below), yet it is not selectively sequentially pseudocompact.
- There exists a countably compact space which is not selectively sequentially pseudocompact (take infinite countaby compact subset of $\beta\omega$).

$$\begin{array}{ccc}
\text{sequentially compact} & \longrightarrow & \text{countably compact} \\
\downarrow & & \downarrow \\
\text{selectively sequentially pseudocompact} & \longrightarrow & \text{selectively pseudocompact} \\
\downarrow & & \downarrow \\
\text{sequentially pseudocompact} & \longrightarrow & \text{pseudocompact}
\end{array}$$

Basic properties of selective (sequential) pseudocompactness

- Suppose that X is a topological space having the following property: For every countable family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists a selectively (sequentially) pseudocompact subspace Y of X such that $U_n \cap Y \neq \emptyset$ for all $n \in \mathbb{N}$. Then X is selectively (sequentially) pseudocompact.
- If some dense subspace of a space X is selectively (sequentially) pseudocompact, then X itself is selectively (sequentially) pseudocompact.
- If every countable subset of a topological space X is contained in a selectively (sequentially) pseudocompact subspace of X, then X is selectively (sequentially) pseudocompact.
- Let $f : X \to Y$ be a continuous function from a topological space X onto a topological space Y. If X is selectively (sequentially) pseudocompact, then so is Y.

Basic properties of selective (sequential) pseudocompactness

- Suppose that X is a topological space having the following property: For every countable family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists a selectively (sequentially) pseudocompact subspace Y of X such that $U_n \cap Y \neq \emptyset$ for all $n \in \mathbb{N}$. Then X is selectively (sequentially) pseudocompact.
- If some dense subspace of a space X is selectively (sequentially) pseudocompact, then X itself is selectively (sequentially) pseudocompact.
- If every countable subset of a topological space X is contained in a selectively (sequentially) pseudocompact subspace of X, then X is selectively (sequentially) pseudocompact.
- Let $f : X \to Y$ be a continuous function from a topological space X onto a topological space Y. If X is selectively (sequentially) pseudocompact, then so is Y.

Basic properties of selective (sequential) pseudocompactness

- Suppose that X is a topological space having the following property: For every countable family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists a selectively (sequentially) pseudocompact subspace Y of X such that $U_n \cap Y \neq \emptyset$ for all $n \in \mathbb{N}$. Then X is selectively (sequentially) pseudocompact.
- If some dense subspace of a space X is selectively (sequentially) pseudocompact, then X itself is selectively (sequentially) pseudocompact.
- If every countable subset of a topological space X is contained in a selectively (sequentially) pseudocompact subspace of X, then X is selectively (sequentially) pseudocompact.
- Let $f : X \to Y$ be a continuous function from a topological space X onto a topological space Y. If X is selectively (sequentially) pseudocompact, then so is Y.

Basic properties of selective (sequential) pseudocompactness

- Suppose that X is a topological space having the following property: For every countable family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists a selectively (sequentially) pseudocompact subspace Y of X such that $U_n \cap Y \neq \emptyset$ for all $n \in \mathbb{N}$. Then X is selectively (sequentially) pseudocompact.
- If some dense subspace of a space X is selectively (sequentially) pseudocompact, then X itself is selectively (sequentially) pseudocompact.
- If every countable subset of a topological space X is contained in a selectively (sequentially) pseudocompact subspace of X, then X is selectively (sequentially) pseudocompact.
- Let $f : X \to Y$ be a continuous function from a topological space X onto a topological space Y. If X is selectively (sequentially) pseudocompact, then so is Y.

selective (sequential) pseudocompactness in (Σ)-products

Definition

If $\{X_i : i \in I\}$ is a family of sets, $X = \prod\{X_i : i \in I\}$ is their product and p is a point in X, then the subset

$$\Sigma(p, X) = \{f \in X : |\{i \in I : f(i) \neq p(i)\}| \leq \omega\}$$

of X is called the Σ*-product of* $\{X_i : i \in I\}$ *with the basis point* $p \in X$.

Theorem

A Σ-product of any family of selectively sequentially pseudocompact spaces is selectively sequentially pseudocompact.

Theorem

If every countable sub-product of the product $X = \prod\{X_i : i \in I\}$ of a family $\{X_i : i \in I\}$ of topological spaces is selectively pseudocompact, then every Σ-product in X is selectively pseudocompact.

Theorem

A Σ-product of any family of selectively sequentially pseudocompact spaces is selectively sequentially pseudocompact.

Theorem

If every countable sub-product of the product $X = \prod\{X_i : i \in I\}$ of a family $\{X_i : i \in I\}$ of topological spaces is selectively pseudocompact, then every Σ-product in X is selectively pseudocompact.

Corollary

A product of topological spaces is selectively sequentially pseudocompact if and only if each factor is selectively sequentially pseudocompact.

Corollary

A product of topological spaces is selectively pseudocompact if and only if all its countable sub-products are selectively pseudocompact.

Example

Let X a countably compact space such that X^2 is not pseudocompact. Since countably compact spaces are selectively pseudocompact, X is selectively pseudocompact. Since X^2 is not even pseudocompact, selective pseudocompactness is not productive.

Corollary

A product of topological spaces is selectively sequentially pseudocompact if and only if each factor is selectively sequentially pseudocompact.

Corollary

A product of topological spaces is selectively pseudocompact if and only if all its countable sub-products are selectively pseudocompact.

Example

Let X a countably compact space such that X^2 is not pseudocompact. Since countably compact spaces are selectively pseudocompact, X is selectively pseudocompact. Since X^2 is not even pseudocompact, selective pseudocompactness is not productive.

selective (sequential) pseudocompactness in topological groups

Theorem (Artico, Marconi, Pelant, Rotter, Tkachenko)

Every pseudocompact group is sequentially pseudocompact.

Definition

A topological space X is *ω-bounded* if the closure of any countable subset is compact.

Theorem

Every ω-bounded group is selectively sequentially pseudocompact. In particular, every compact group is selectively sequentially pseudocompact.

selective (sequential) pseudocompactness in topological groups

Theorem (Artico, Marconi, Pelant, Rotter, Tkachenko)

Every pseudocompact group is sequentially pseudocompact.

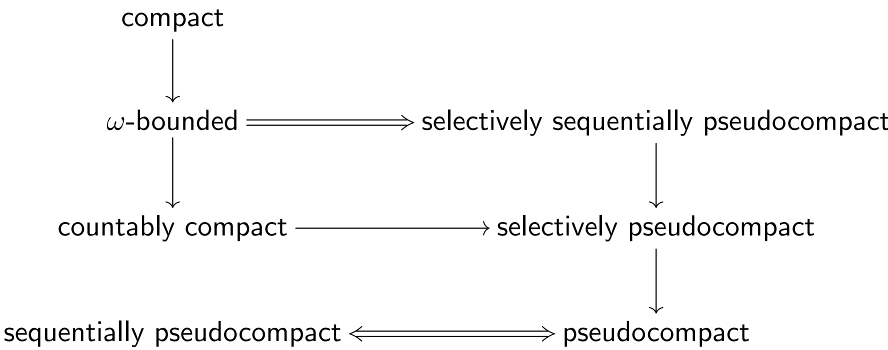
Definition

A topological space X is *ω-bounded* if the closure of any countable subset is compact.

Theorem

Every ω-bounded group is selectively sequentially pseudocompact. In particular, every compact group is selectively sequentially pseudocompact.

In the class of topological groups, we have the following diagram.



We now concentrate on two implications:

sel. seq. pseudocompact $\to$ sel. pseudocompact $\to$ pseudocompact

- An example of a pseudocompact Abelian group which is not selectively pseudocompact was constructed by García-Ferreira and Tomita.
- If G is a countably compact Abelian group without non-trivial convergent sequences, then G is sequentially pseudocompact but not selectively sequentially pseudocompact. (Such groups exist under CH, for example.)
- We do not yet have a ZFC example of a selectively pseudocompact group which is not selectively sequentially pseudocompact, but we are presently working hard on getting it!

We now concentrate on two implications:

sel. seq. pseudocompact $\to$ sel. pseudocompact $\to$ pseudocompact

- An example of a pseudocompact Abelian group which is not selectively pseudocompact was constructed by García-Ferreira and Tomita.
- If G is a countably compact Abelian group without non-trivial convergent sequences, then G is sequentially pseudocompact but not selectively sequentially pseudocompact. (Such groups exist under CH, for example.)
- We do not yet have a ZFC example of a selectively pseudocompact group which is not selectively sequentially pseudocompact, but we are presently working hard on getting it!

We now concentrate on two implications:

sel. seq. pseudocompact $\rightarrow$ sel. pseudocompact $\rightarrow$ pseudocompact

- An example of a pseudocompact Abelian group which is not selectively pseudocompact was constructed by García-Ferreira and Tomita.
- If G is a countably compact Abelian group without non-trivial convergent sequences, then G is sequentially pseudocompact but not selectively sequentially pseudocompact. (Such groups exist under CH, for example.)
- We do not yet have a ZFC example of a selectively pseudocompact group which is not selectively sequentially pseudocompact, but we are presently working hard on getting it!

We now concentrate on two implications:

sel. seq. pseudocompact $\rightarrow$ sel. pseudocompact $\rightarrow$ pseudocompact

- An example of a pseudocompact Abelian group which is not selectively pseudocompact was constructed by García-Ferreira and Tomita.
- If G is a countably compact Abelian group without non-trivial convergent sequences, then G is sequentially pseudocompact but not selectively sequentially pseudocompact. (Such groups exist under CH, for example.)
- We do not yet have a ZFC example of a selectively pseudocompact group which is not selectively sequentially pseudocompact, but we are presently working hard on getting it!

sel. seq. pseudocompact → sel. pseudocompact → pseudocompact

García-Ferreira and Tomita asked if the last implication can be reversed when the existential quantifier is added to both sides of it.

Problem (García-Ferreira and Tomita)

If an Abelian group admits a pseudocompact group topology, does it admit a selectively pseudocompact group topology?

One can also consider the following stronger version of this problem:

Problem

If an Abelian group admits a pseudocompact group topology, does it admit a selectively sequentially pseudocompact group topology?

In the rest of this talk, we provide a positive answer to these questions for large classes of abelian groups.

sel. seq. pseudocompact $\to$ sel. pseudocompact $\to$ pseudocompact

García-Ferreira and Tomita asked if the last implication can be reversed when the existential quantifier is added to both sides of it.

Problem (García-Ferreira and Tomita)

If an Abelian group admits a pseudocompact group topology, does it admit a selectively pseudocompact group topology?

One can also consider the following stronger version of this problem:

Problem

If an Abelian group admits a pseudocompact group topology, does it admit a selectively sequentially pseudocompact group topology?

In the rest of this talk, we provide a positive answer to these questions for large classes of abelian groups.

sel. seq. pseudocompact $\rightarrow$ sel. pseudocompact $\rightarrow$ pseudocompact

García-Ferreira and Tomita asked if the last implication can be reversed when the existential quantifier is added to both sides of it.

Problem (García-Ferreira and Tomita)

If an Abelian group admits a pseudocompact group topology, does it admit a selectively pseudocompact group topology?

One can also consider the following stronger version of this problem:

Problem

If an Abelian group admits a pseudocompact group topology, does it admit a selectively sequentially pseudocompact group topology?

In the rest of this talk, we provide a positive answer to these questions for large classes of abelian groups.

Our results depend on some additional set-theoretic assumptions which we outline below.

Let τ be a cardinal. As usual, symbol τ^ω denotes the cardinality of the set $[X]^\omega$ of all countably infinite subsets of some (equivalently, every) set X such that $|X| = \sigma$, while 2^τ denotes the cardinality of the set of all subsets of some (equivalently, every) set X such that $|X| = \tau$. Symbol $\mathrm{cf}(\tau)$ denotes the *cofinality* of τ, i.e. the smallest cardinal λ for which there exists a representation $\tau = \Sigma\{\tau_\alpha : \alpha < \lambda\}$ where $\tau_\alpha < \tau$ for all $\alpha < \lambda$. Finally, τ^+ denotes the smallest cardinal bigger than τ.

We fix the symbol SCH for denoting the following condition on cardinals:

$$\textit{if } \tau \geq \mathfrak{c} \textit{ is a cardinal and } \mathrm{cf}(\tau) \neq \omega, \textit{ then } \tau^\omega = \tau. \qquad (1)$$

We choose the abbreviation symbol SCH for denoting the condition (1) since it is known to be equivalent to the Singular Cardinals Hypothesis.

Our results depend on some additional set-theoretic assumptions which we outline below.

Let τ be a cardinal. As usual, symbol τ^ω denotes the cardinality of the set $[X]^\omega$ of all countably infinite subsets of some (equivalently, every) set X such that $|X| = \sigma$, while 2^τ denotes the cardinality of the set of all subsets of some (equivalently, every) set X such that $|X| = \tau$. Symbol $\mathrm{cf}(\tau)$ denotes the *cofinality* of τ, i.e. the smallest cardinal λ for which there exists a representation $\tau = \Sigma\{\tau_\alpha : \alpha < \lambda\}$ where $\tau_\alpha < \tau$ for all $\alpha < \lambda$. Finally, τ^+ denotes the smallest cardinal bigger than τ.

We fix the symbol SCH for denoting the following condition on cardinals:

$$\textit{if } \tau \geq \mathfrak{c} \textit{ is a cardinal and } \mathrm{cf}(\tau) \neq \omega, \textit{ then } \tau^\omega = \tau. \tag{1}$$

We choose the abbreviation symbol SCH for denoting the condition (1) since it is known to be equivalent to the Singular Cardinals Hypothesis.

The Generalized Continuum Hypothesis (abbreviated GCH) states that $2^\tau = \tau^+$ for every infinite cardinal τ. It is well known that GCH implies SCH and that GCH is consistent with the usual axioms ZFC of set theory.

Prikry and Silver have constructed models of set theory violating SCH assuming the consistency of supercompact cardinals. Devlin and Jensen proved that the failure of SCH implies the existence of an inner model with a large cardinal. Therefore, large cardinals are needed in order to violate SCH, and so SCH can be viewed as a "mild" additional set-theoretic assumption beyond the axioms of ZFC.

The Generalized Continuum Hypothesis (abbreviated GCH) states that $2^\tau = \tau^+$ for every infinite cardinal τ. It is well known that GCH implies SCH and that GCH is consistent with the usual axioms ZFC of set theory.

Prikry and Silver have constructed models of set theory violating SCH assuming the consistency of supercompact cardinals. Devlin and Jensen proved that the failure of SCH implies the existence of an inner model with a large cardinal. Therefore, large cardinals are needed in order to violate SCH, and so SCH can be viewed as a "mild" additional set-theoretic assumption beyond the axioms of ZFC.

The Generalized Continuum Hypothesis (abbreviated GCH) states that $2^\tau = \tau^+$ for every infinite cardinal τ. It is well known that GCH implies SCH and that GCH is consistent with the usual axioms ZFC of set theory.
Prikry and Silver have constructed models of set theory violating SCH assuming the consistency of supercompact cardinals. Devlin and Jensen proved that the failure of SCH implies the existence of an inner model with a large cardinal. Therefore, large cardinals are needed in order to violate SCH, and so SCH can be viewed as a "mild" additional set-theoretic assumption beyond the axioms of ZFC.

selectively sequentially pseudocompact group topologies on $\mathcal{V}$-free groups

Recall that a *variety of groups* is a class of (abstract) groups closed under Cartesian products, subgroups and quotients.

Definition

Let $\mathcal{V}$ be a variety of groups. A subset X of a group G is said to be:

(i) *$\mathcal{V}$-independent* if

1. $\langle X \rangle \in \mathcal{V}$, and
2. for each map $f : X \to H \in \mathcal{V}$ there exists a unique homomorphism $\tilde{f} : \langle X \rangle \to H$ extending f.

(ii) a *$\mathcal{V}$-base* of G if X is $\mathcal{V}$-independent and $\langle X \rangle = G$.

For a subset X of a group G, the symbol $\langle X \rangle$ denotes the smallest subgroup of G containing X.

selectively sequentially pseudocompact group topologies on $\mathcal{V}$-free groups

Recall that a *variety of groups* is a class of (abstract) groups closed under Cartesian products, subgroups and quotients.

Definition

Let $\mathcal{V}$ be a variety of groups. A subset X of a group G is said to be:

(i) $\mathcal{V}$-*independent* if

1. $\langle X \rangle \in \mathcal{V}$, and
2. for each map $f : X \to H \in \mathcal{V}$ there exists a unique homomorphism $\tilde{f} : \langle X \rangle \to H$ extending f.

(ii) a $\mathcal{V}$-*base* of G if X is $\mathcal{V}$-independent and $\langle X \rangle = G$.

For a subset X of a group G, the symbol $\langle X \rangle$ denotes the smallest subgroup of G containing X.

Definition

Let $\mathcal{V}$ be a variety of groups. A subset X of a group G is said to be:

(i) *$\mathcal{V}$-independent* if

1. $\langle X \rangle \in \mathcal{V}$, and
2. for each map $f : X \to H \in \mathcal{V}$ there exists a unique homomorphism $\tilde{f} : \langle X \rangle \to H$ extending f.

(ii) a *$\mathcal{V}$-base* of G if X is $\mathcal{V}$-independent and $\langle X \rangle = G$.

For a subset X of a group G, the symbol $\langle X \rangle$ denotes the smallest subgroup of G containing X.

Definition

Let $\mathcal{V}$ be a non-trivial variety of groups.

(i) A group is *$\mathcal{V}$-free* if it contains a $\mathcal{V}$-base.

(ii) For every cardinal τ, we denote by $F_\tau(\mathcal{V})$ the unique (up to isomorphism) $\mathcal{V}$-free group having a $\mathcal{V}$-base of cardinality τ.

Definition (Dikranjan, Shakhmatov)

A variety $\mathcal{V}$ is called *precompact* provided that, for every positive integer n, there exists a compact group $H_n \in \mathcal{V}$ containing a $\mathcal{V}$-independent subset of size n.

Non-precompact varieties are not easy to come by, as many of the known varieties are precompact; this includes all abelian varieties, all nilpotent groups, all polynilpotent groups, all soluble groups. For any prime number $p > 665$, the Burnside variety $\mathcal{B}_p$ consisting of all groups satisfying the identity $x^p = e$ is not precompact (Dikranjan).

Precompactness of a variety $\mathcal{V}$ is a necessary condition for the existence of a pseudocomact group topology on any $\mathcal{V}$-free group with infinitely many generators (Dikranjan, Shakhmatov).

Definition (Dikranjan, Shakhmatov)

A variety $\mathcal{V}$ is called *precompact* provided that, for every positive integer n, there exists a compact group $H_n \in \mathcal{V}$ containing a $\mathcal{V}$-independent subset of size n.

Non-precompact varieties are not easy to come by, as many of the known varieties are precompact; this includes all abelian varieties, all nilpotent groups, all polynilpotent groups, all soluble groups.

For any prime number $p > 665$, the Burnside variety $\mathcal{B}_p$ consisting of all groups satisfying the identity $x^p = e$ is not precompact (Dikranjan).

Precompactness of a variety $\mathcal{V}$ is a necessary condition for the existence of a pseudocomact group topology on any $\mathcal{V}$-free group with infinitely many generators (Dikranjan, Shakhmatov).

Definition (Dikranjan, Shakhmatov)

A variety $\mathcal{V}$ is called *precompact* provided that, for every positive integer n, there exists a compact group $H_n \in \mathcal{V}$ containing a $\mathcal{V}$-independent subset of size n.

Non-precompact varieties are not easy to come by, as many of the known varieties are precompact; this includes all abelian varieties, all nilpotent groups, all polynilpotent groups, all soluble groups. For any prime number $p > 665$, the Burnside variety $\mathcal{B}_p$ consisting of all groups satisfying the identity $x^p = e$ is not precompact (Dikranjan).

Precompactness of a variety $\mathcal{V}$ is a necessary condition for the existence of a pseudocomact group topology on any $\mathcal{V}$-free group with infinitely many generators (Dikranjan, Shakhmatov).

Definition (Dikranjan, Shakhmatov)

A variety $\mathcal{V}$ is called *precompact* provided that, for every positive integer n, there exists a compact group $H_n \in \mathcal{V}$ containing a $\mathcal{V}$-independent subset of size n.

Non-precompact varieties are not easy to come by, as many of the known varieties are precompact; this includes all abelian varieties, all nilpotent groups, all polynilpotent groups, all soluble groups. For any prime number $p > 665$, the Burnside variety $\mathcal{B}_p$ consisting of all groups satisfying the identity $x^p = e$ is not precompact (Dikranjan).

Precompactness of a variety $\mathcal{V}$ is a necessary condition for the existence of a pseudocomact group topology on any $\mathcal{V}$-free group with infinitely many generators (Dikranjan, Shakhmatov).

Theorem

Let $\mathcal{V}$ be a precompact variety of groups. Then for every infinite cardinal σ such that $\sigma^\omega = \sigma$, the group $F_\sigma(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Corollary

Let $\mathcal{V}$ be a precompact variety of groups. Under GCH, the following conditions are equivalent for every infinite cardinal σ:

(i) *the group $F_\sigma(\mathcal{V})$ admits a pseudocompact group topology;*
(ii) *the group $F_\sigma(\mathcal{V})$ admits a selectively pseudocompact group topology;*
(iii) *the group $F_\sigma(\mathcal{V})$ admits a selectively sequentially pseudocompact group topology;*
(iv) *the group $F_\sigma(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.*

Theorem

Let $\mathcal{V}$ be a precompact variety of groups. Then for every infinite cardinal σ such that $\sigma^\omega = \sigma$, the group $F_\sigma(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Corollary

Let $\mathcal{V}$ be a precompact variety of groups. Under GCH, the following conditions are equivalent for every infinite cardinal σ:

(i) *the group $F_\sigma(\mathcal{V})$ admits a pseudocompact group topology;*

(ii) *the group $F_\sigma(\mathcal{V})$ admits a selectively pseudocompact group topology;*

(iii) *the group $F_\sigma(\mathcal{V})$ admits a selectively sequentially pseudocompact group topology;*

(iv) *the group $F_\sigma(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.*

We use $\mathcal{G}$ for denoting the variety of all groups.

Theorem

For every infinite cardinal σ such that $\sigma^\omega = \sigma$, the group $F_\sigma(\mathcal{G})$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Corollary

Under GCH, the following conditions are equivalent for every infinite cardinal σ:

(i) $F_\sigma(\mathcal{G})$ *admits a pseudocompact group topology;*

(ii) $F_\sigma(\mathcal{G})$ *admits a selectively pseudocompact group topology;*

(iii) $F_\sigma(\mathcal{G})$ *admits a selectively sequentially pseudocompact group topology;*

(iv) $F_\sigma(\mathcal{G})$ *admits a zero-dimensional selectively sequentially pseudocompact group topology;*

(v) $F_\sigma(\mathcal{G})$ *admits a connected, locally connected, selectively sequentially pseudocompact group topology.*

Theorem

For every infinite cardinal σ such that $\sigma^\omega = \sigma$, the group $F_\sigma(\mathcal{G})$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Corollary

Under GCH, the following conditions are equivalent for every infinite cardinal σ:

(i) $F_\sigma(\mathcal{G})$ *admits a pseudocompact group topology;*

(ii) $F_\sigma(\mathcal{G})$ *admits a selectively pseudocompact group topology;*

(iii) $F_\sigma(\mathcal{G})$ *admits a selectively sequentially pseudocompact group topology;*

(iv) $F_\sigma(\mathcal{G})$ *admits a zero-dimensional selectively sequentially pseudocompact group topology;*

(v) $F_\sigma(\mathcal{G})$ *admits a connected, locally connected, selectively sequentially pseudocompact group topology.*

Definition

We shall say that a cardinal τ is *selectively admissible* if there exists an infinite cardinal σ such that $\sigma^\omega = \sigma \leq \tau \leq 2^\sigma$.

We use $\mathcal{A}$ for denoting the varietiey of all Abelian groups. For $n \in \mathbb{N}$ we denote by $\mathcal{B}_n$ the Burnside variety of all groups satisfying the identity $x^n = e$. Any proper sub-variety of $\mathcal{A}$ coincides with the variety $\mathcal{A}_n = \mathcal{B}_n \cap \mathcal{A}$ for an appropriate $n \in \mathbb{N}$.

Theorem

Let $\mathcal{V}$ be a subvariety of the variety $\mathcal{A}$ of all Abelian groups. Then for every selectively admissible cardinal τ, the $\mathcal{V}$-free group $F_\tau(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Definition

We shall say that a cardinal τ is *selectively admissible* if there exists an infinite cardinal σ such that $\sigma^\omega = \sigma \leq \tau \leq 2^\sigma$.

We use $\mathcal{A}$ for denoting the varietiey of all Abelian groups. For $n \in \mathbb{N}$ we denote by $\mathcal{B}_n$ the Burnside variety of all groups satisfying the identity $x^n = e$. Any proper sub-variety of $\mathcal{A}$ coincides with the variety $\mathcal{A}_n = \mathcal{B}_n \cap \mathcal{A}$ for an appropriate $n \in \mathbb{N}$.

Theorem

Let $\mathcal{V}$ be a subvariety of the variety $\mathcal{A}$ of all Abelian groups. Then for every selectively admissible cardinal τ, the $\mathcal{V}$-free group $F_\tau(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Definition

We shall say that a cardinal τ is *selectively admissible* if there exists an infinite cardinal σ such that $\sigma^\omega = \sigma \leq \tau \leq 2^\sigma$.

We use $\mathcal{A}$ for denoting the varietiey of all Abelian groups. For $n \in \mathbb{N}$ we denote by $\mathcal{B}_n$ the Burnside variety of all groups satisfying the identity $x^n = e$. Any proper sub-variety of $\mathcal{A}$ coincides with the variety $\mathcal{A}_n = \mathcal{B}_n \cap \mathcal{A}$ for an appropriate $n \in \mathbb{N}$.

Theorem

Let $\mathcal{V}$ be a subvariety of the variety $\mathcal{A}$ of all Abelian groups. Then for every selectively admissible cardinal τ, the $\mathcal{V}$-free group $F_\tau(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.

Corollary

Let $\mathcal{V}$ be a subvariety of the variety $\mathcal{A}$ of all Abelian groups. Under SCH, the following conditions are equivalent for every infinite cardinal τ:

(i) *$F_\tau(\mathcal{V})$ admits a pseudocompact group topology;*

(ii) *$F_\tau(\mathcal{V})$ admits a selectively pseudocompact group topology;*

(iii) *$F_\tau(\mathcal{V})$ admits a selectively sequentially pseudocompact group topology;*

(iv) *$F_\tau(\mathcal{V})$ admits a zero-dimensional selectively sequentially pseudocompact group topology.*

Theorem

For every selectively admissible cardinal τ, the $\mathcal{A}$-free group $F_\tau(\mathcal{A})$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Corollary

Under SCH, the following conditions are equivalent for every infinite cardinal τ:

(i) $F_\tau(\mathcal{A})$ *admits a pseudocompact group topology;*

(ii) $F_\tau(\mathcal{A})$ *admits a selectively pseudocompact group topology;*

(iii) $F_\tau(\mathcal{A})$ *admits a selectively sequentially pseudocompact group topology;*

(iv) $F_\tau(\mathcal{A})$ *admits a zero-dimensional selectively sequentially pseudocompact group topology;*

(v) $F_\tau(\mathcal{A})$ *admits a connected, locally connected, selectively sequentially pseudocompact group topology.*

Theorem

For every selectively admissible cardinal τ, the $\mathcal{A}$-free group $F_\tau(\mathcal{A})$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.

Corollary

Under SCH, the following conditions are equivalent for every infinite cardinal τ:

(i) *$F_\tau(\mathcal{A})$ admits a pseudocompact group topology;*
(ii) *$F_\tau(\mathcal{A})$ admits a selectively pseudocompact group topology;*
(iii) *$F_\tau(\mathcal{A})$ admits a selectively sequentially pseudocompact group topology;*
(iv) *$F_\tau(\mathcal{A})$ admits a zero-dimensional selectively sequentially pseudocompact group topology;*
(v) *$F_\tau(\mathcal{A})$ admits a connected, locally connected, selectively sequentially pseudocompact group topology.*

Torsion groups

Theorem

Under SCH, the following conditions are equivalent for every torsion Abelian group G:

(i) *G has a pseudocompact group topology;*

(ii) *G has a selectively pseudocompact group topology;*

(iii) *G has a selectively sequentially pseudocompact group topology.*

Torsion-free groups

Theorem

Under SCH, the following conditions are equivalent for every torsion-free Abelian group G:

(i) *G admits a pseudocompact group topology;*

(ii) *G admits a selectively pseudocompact group topology;*

(iii) *G admits a selectively sequentially pseudocompact group topology;*

(iv) *G admits a connected, locally connected, selectively sequentially pseudocompact group topology.*

This theorem remains valid for a larger class of Abelian groups G satisfying $r_{\mathcal{A}}(G) = |G|$, where

$$r_{\mathcal{A}}(G) = \sup\{|X| : X \text{ is a } \mathcal{A}\text{-independent subset of } G\}$$

is the $\mathcal{A}$-rank of G.

Torsion-free groups

Theorem

Under SCH, the following conditions are equivalent for every torsion-free Abelian group G:

(i) *G admits a pseudocompact group topology;*

(ii) *G admits a selectively pseudocompact group topology;*

(iii) *G admits a selectively sequentially pseudocompact group topology;*

(iv) *G admits a connected, locally connected, selectively sequentially pseudocompact group topology.*

This theorem remains valid for a larger class of Abelian groups G satisfying $r_{\mathcal{A}}(G) = |G|$, where

$$r_{\mathcal{A}}(G) = \sup\{|X| : X \text{ is a } \mathcal{A}\text{-independent subset of } G\}$$

is the $\mathcal{A}$-rank of G.

Connected selectively sequentially pseudocompact group topologies

Theorem

Under SCH, the following conditions are equivalent for every Abelian group G:

(i) *G admits a connected pseudocompact group topology;*

(ii) *G admits a connected selectively pseudocompact group topology;*

(iii) *G admits a connected, selectively sequentially pseudocompact group topology;*

(iv) *G admits a connected, locally connected, selectively sequentially pseudocompact group topology.*

selectively sequentially pseudocompact group topologies on divisible groups

Since every pseudocompact group topology on a divisible Abelian group is automatically connected, we get the following

Corollary

Under SCH, the following conditions are equivalent for every divisible Abelian group G:

(i) *G admits a pseudocompact group topology;*

(ii) *G admits a selectively pseudocompact group topology;*

(iii) *G admits a selectively sequentially pseudocompact group topology;*

(iv) *G admits a (connected and) locally connected, selectively sequentially pseudocompact group topology.*

selectively sequentially pseudocompact group topologies which are not countably compact

Example

Let G be the $\mathcal{G}$-free group of size $\mathfrak{c}$. By our theorem, *G admits a selectively sequentially pseudocompact group topology*. On the other hand, *no non-trivial subgroup of G admits a countably compact group topology*. Indeed, if H is a subgroup of G, then it is $\mathcal{G}$-free, and it is known that no non-trivial $\mathcal{G}$-free group admits a countably compact group topology. In particular, *there exists a selectively sequentially pseudocompact group without non-trivial countably compact subgroups*.

Example

Let $G = F_{\mathcal{A}}(\mathfrak{c})^\omega$ be the countable power of the $\mathcal{A}$-free group of size $\mathfrak{c}$. Then G is an Abelian group of size $\mathfrak{c}$. By our theorem,$F_{\mathcal{A}}(\mathfrak{c})$ admits a selectively sequentially pseudocompact group topology. Since the class of selectively sequentially pseudocompact spaces is closed under arbitrary products, G also admits a selectively sequentially pseudocompact group topology. By Tomita's theorem, this topology cannot be countably compact. Therefore, *G is an Abelian group (of size $\mathfrak{c}$) which admits a selectively sequentially pseudocompact group topology, yet does not admit a countably compact group topology.*