# On Corson and Valdivia compact spaces*

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In this talk we deal with several classes of nonmetrizable compact spaces that correspond to well-known classes of Banach spaces with many projections. In particular, we discuss the class of Valdivia compact spaces and its subclass of Corson compact spaces.

Let $I = [0,1]$. Given a set $A$, the $\Sigma$-product of the product $I^A$ is the set

$$\Sigma I^A := \{f \in I^A : \left|f^{-1}((0,1])\right| \leq \omega\}.$$

Definition

- A set $Y \subset X$ will be called a **$\Sigma$-subset** of $X$ if there is an embedding $\phi : X \to I^A$, for some set $A$, such that

$$Y = \phi^{-1}(\phi(X) \cap \Sigma I^A).$$

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Kubiś and Michalewski investigated a $\sigma$-complete inverse system whose bonding mappings are retractions and use it to obtain a characterization of Valdivia compact spaces.
From now on, $\Gamma$ will denote an up-directed $\sigma$-complete partially ordered set.

Definition (Kubiś and Michalewski, 2006)

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(i) $r_s(X)$ is cosmic for each $s \in \Gamma$.

(ii) $r_s = r_s \circ r_t = r_t \circ r_s$ whenever $s \leq t$.

(iii) If $s \in \Gamma$ and $s = \sup_{n\in\mathbb{N}} s_n \uparrow$, then $r_s = \lim_{n\to\infty} r_{s_n}$.

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## A characterization of Valdivia compacta

An $r$-skeleton $\{r_s : s \in \Gamma\}$ on $X$ is **commutative** if $r_s \circ r_t = r_t \circ r_s$ for every $s, t \in \Gamma$.

Theorem (Kubiś and Michalewski, 2006)

*A compact space $X$ is Valdivia if and only if admits a commutative $r$-skeleton.*

This characterization was used to prove that a compact space of weight $\omega_1$ is Valdivia compact iff it is the limit of an inverse sequence of metric compacta whose bonding maps are retractions. As a corollary, it was proved that the class of Valdivia compacta of weight $\omega_1$ is preserved both under retractions and under open 0-dimensional images.

Theorem (Chigogidze, 2008)

*Let $X$ be a compact group. Then $X$ is a Valdivia compact iff $X$ is homeomorphic to a product of metrizable compacta.*

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Theorem (Cúth, 2014)

*A compact space $X$ is Corson if and only if admits a full $r$-skeleton.*

Theorem (Bandlow, 1991)

*Let $K$ be a compact space. Then $K$ is Corson iff, for every large enough cardinal $\theta$, there exists a closed and unbounded family $\mathcal{C} \subset [H(\theta)]^{\leq\omega}$ of elementary substructures $(H(\theta), \in)$ such that for each $M \in \mathcal{C}$ the quotient map $\Delta(C(X) \cap M) : K \to \mathbb{R}^{C(X)\cap M}$ is one-to-one on $\overline{K \cap M}$.*

It is natural to try to get a proof of the characterization of Valdivia compact spaces by using Bandlow's ideas.

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The $r$-skeletons in compact and countably compact spaces have several nice properties.

**Lemma**

*Let $X$ be a countably compact space $X$. If $\{r_s : s \in \Gamma\}$ is a family of retractions in a $X$ satisfying* (i) - (iii) *from the definition of $r$-skeleton. If $Y = \bigcup\{r_s(X) : s \in \Gamma\}$, then*

- $t(Y) \leq \omega$.
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**Lemma**

*Let $X$ be a compact space and let $F$ be closed in $X$. Suppose that $\{r_s : s \in \Gamma\}$ is a family of retractions from $X$ into $F$ such that $\{r_s \restriction_F : s \in \Gamma\}$ is an $r$-skeleton on $F$. If $R = \Delta\{r_s \restriction_F : s \in \Gamma\}$, then $R \restriction_F : F \to R(X)$ is a homeomorphism.*

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*Let $X$ be a compact space and let $F$ be closed in $X$. Suppose that $\{r_s : s \in \Gamma\}$ is a family of retractions from $X$ into $F$ such that $\{r_s \restriction_F : s \in \Gamma\}$ is an $r$-skeleton on $F$. If $R = \Delta\{r_s \restriction_F : s \in \Gamma\}$, then $R \restriction_F : F \to R(X)$ is a homeomorphism.*

The $r$-skeletons in compact and countably compact spaces have several nice properties.

### Lemma

*Let $X$ be a countably compact space $X$. If $\{r_s : s \in \Gamma\}$ is a family of retractions in a $X$ satisfying* (i) - (iii) *from the definition of $r$-skeleton. If $Y = \bigcup\{r_s(X) : s \in \Gamma\}$, then*

- $t(Y) \leq \omega$.
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### Lemma

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*Let $X$ be compact and let $Y$ be induced by a commutative $r$-skeleton. Then there exists a family $\{r_A : A \in \mathcal{P}(Y)\}$ of retractions on $X$ such that, if $X_A = r_A(X)$ then:*

(i) *The family $\{r_B : B \in [Y]^{\leq\omega}\}$ is a commutative $r$-skeleton on $X_A$ and induces $Y \cap X_A$.*

(ii) *$A \subset X_A$ and $d(X_A) \leq |A|$.*

(iii) *$r_B \circ r_A = r_A \circ r_B = r_B$ whenever $B \subset A$.*

(iv) *If $A = \bigcup_{\alpha<\lambda} A_\alpha \uparrow \in \mathcal{P}(Y)$ then $r_A = \lim r_{A_\alpha}$.*

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To prove that result we get an $r$-skeleton $\{r_A : A \in [Y]^{\leq\omega}\}$ satisfying (ii) and use the previous two Lemmas.

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Theorem

*Let $Y$ be a dense subspace of a compact space $X$. If $Y$ is induced by a commutative r-skeleton in $X$, then $Y$ is a $\Sigma$-subset of $X$.*

**Proof.** By induction on the density of $Y$. Assume that $d(Y) = \kappa > \omega$ and the result holds for spaces of density at most $\kappa$. Choose a family $\{r_A : A \in \mathcal{P}(X)\}$ of retractions in $X$ as in the last Lemma. Let $\{y_\alpha : \alpha < \kappa\}$ be a dense subspace of $Y$. For each $\alpha \leq \kappa$, set $A_\alpha = \{x_\beta : \beta < \alpha\}$, $r_\alpha = r_{A_\alpha}$ and $X_\alpha = r_\alpha(X)$. Given $\alpha < \kappa$ we can find a set $T_\alpha$ and an embedding $\phi_\alpha : X_\alpha \to \mathbb{R}^{T_\alpha}$ such that $Y \cap X_\alpha = \phi_\alpha^{-1}(\Sigma\mathbb{R}^{T_\alpha})$. Let $T = \bigcup\{T_\alpha : \alpha < \kappa\}$. Define $\phi : X \to \mathbb{R}^T$ as follows: If $x \in X$ and $\alpha < \kappa$, we set

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Corollary

*A compact space $X$ is Valdivia if and only if admits a commutative $r$-skeleton.*

It happens that the proof also works for the case of Corson compact spaces.

Corollary

*A compact space $X$ is Corson iff and only if admits a full $r$-skeleton.*

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*If a countably compact space, $X$ has a full $r$-skeleton and has weight at most $\omega_1$, then $X$ can be embedded in a $\Sigma\mathbb{R}^{\omega_1}$.*

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Bandlow uses his result to obtain a characterization of the space $C_p(X)$ for a Corson compact space $X$. It is natural to ask if there exists a similar characterization in the context of $r$-skeletons.

The next technical notion sometimes result useful.

Definition

A map $\phi : \Gamma \to [Y]^{\leq\omega}$ is called $\omega$-*monotone* provided that:

(a) if $s, t \in \Gamma$ and $s \leq t$, then $\phi(s) \subseteq \phi(t)$.

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Definition

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(i) The set $q_s(D_s)$ is dense in $X_s$.

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Theorem

*If $X$ has a full $q$-skeleton, then every countably compact subspace of $C_p(X)$ has a full $r$-skeleton. In particular, every compact subspace of $C_p(X)$ is Corson.*

Theorem

*If $X$ is monotonically $\omega$-stable, then $X$ has a full $q$-skeleton. In particular, whenever $X$ is either Lindelöf $\Sigma$ or pseudocompact.*

Theorem

*If $K$ is compact and $X$ is a closed subspace of $(L_\kappa)^\omega \times K$, then $X$ has a full $q$-skeleton.*

Corollary (Bandlow, 1994)

*Let $K$ and $X$ be compact; suppose that $C_p(X)$ is a continuous image of a closed subspace of $(L_\kappa)^\omega \times K$. Then $X$ is Corson.*

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Let us observe that all the elements in the definition of $q$-skeleton are dualizable. In this way, it is natural to define a dual concept.

**Definition**

A $c$-**skeleton** on $X$ is a family of pairs $\{(F_s, \mathcal{B}_s) : s \in \Gamma\}$, where $F_s$ is a closed in $X$ and $\mathcal{B}_s \in [\tau(X)]^{\leq\omega}$ for each $s \in \Gamma$, which satisfy:

(i) for each $s \in \Gamma$, $\mathcal{B}_s$ is a base for a topology on $\tau_s$ on $X$ and there exist a Tychonoff space $Z_s$ and a continuous map $g_s : (X, \tau_s) \to Z_s$ which separates the points of $F_s$,

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## Some properties of $c$-skeletons

**Theorem**

*If $X$ has a (full) c-skeleton, then $C_p(X)$ has a (full) q-skeleton.*

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**Corollary**

*A compact space $X$ is Corson iff has a full c-skeleton.*

**Question**

*Let $X$ be a countably compact space, is it true $X$ has a full c-skeleton iff $X$ has a full $r$-skeleton.*

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Consider the following game $G(H, X)$ of length $\omega$ played in a space $X$, where $H$ is a closed subset of $X$. There are two players, $\mathscr{O}$ and $\mathscr{P}$.

- In the nth round, $\mathscr{O}$ chooses an open superset $O_n$ of $H$, and $\mathscr{P}$ chooses a point $p_n \in O_n$.

The player $\mathscr{O}$ wins the game if $p_n \to H$. We say that $H$ is a **$W$-set** in $X$ if $\mathscr{O}$ has a winning strategy for $G(H, X)$.

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*Let $X$ be a countably compact which admits a full $r$-skeleton. If $H$ is non-empty and closed in $X$ then $H$ is a $W$-set in $X$.*

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Definition (J. Bell, 2014)

The proximal game $Prox_{D,P}(X)$ of length $\omega$ played on a uniform space $X$ with two players $\mathscr{D}$, $\mathscr{P}$ proceeds as follows:

- In the initial round 0, $\mathscr{D}$ chooses an open symmetric entourage $D_0$, followed by $\mathscr{P}$ choosing a point $p_0 \in X$.
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At the conclusion of the game, the player $\mathscr{D}$ wins if either $\bigcap\{D_n[p_n] : n \in \omega\} = \emptyset$ or $\{p_n : n \in \mathbb{N}\}$ converges, and $\mathscr{P}$ wins otherwise.

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**Theorem (Clontz and Gruenhague, 2015)**

*All proximal spaces are $W$-spaces.*

**Theorem**

*Let $X$ be a countably compact which admits a full $r$-skeleton. Then $X$ is proximal.*

For countably compact spaces we have:

$$r\text{-skeleton} \longrightarrow \text{Proximal} \longrightarrow W\text{-space}$$

**Question**

*Are the above implications reversible?*

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Given a space $X$, a subspace $Y$ of $X$ is **monotonically retractable** in $X$ if we can assign to each $A \in [Y]^{\leq\omega}$ a retraction $r_A : X \to Y$ and a family $\mathcal{N}(A) \in [\mathcal{P}(Y)]^{\leq\omega}$ such that:

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