Fuzzy uniform structures

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Definition

Let X be a nonempty set. A gauge or a uniform structure on X is a nonempty family $\mathcal{D}$ of pseudometrics on X such that:

(G1) if $d, q \in \mathcal{D}$ then $d \vee q \in \mathcal{D}$;

(G2) if e is a pseudometric on X and for each $\varepsilon > 0$ there exist $d \in \mathcal{D}$ and $\delta > 0$ such that $d(x, y) < \delta$ implies $e(x, y) < \varepsilon$ for all $x, y \in X$, then $e \in \mathcal{D}$.

Definition

A function $f : (X, \mathcal{D}) \to (Y, \mathcal{Q})$ between two spaces endowed with a uniform structure is uniformly continuous if $f : (X, \bigvee_{d \in \mathcal{D}} \mathcal{U}_d) \to (Y, \bigvee_{q \in \mathcal{Q}} \mathcal{U}_q)$ is uniformly continuous

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$$\mathsf{Unif} \underset{\Lambda}{\overset{\Delta}{\rightleftarrows}} \mathsf{SUnif}$$

- $\Delta(\mathcal{U}) = \{d \text{ pseudometric } : \mathcal{U}_d \subseteq \mathcal{U}\}$;
- $\Lambda(\mathcal{D}) = \bigvee_{d \in \mathcal{D}} \mathcal{U}_d$.

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Definition

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with unit 1, such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Example

- $a \wedge b = \min\{a, b\}$
- $a \cdot b = ab$
- $a *_L b = \max\{a + b - 1, 0\}$

$$* \leq \wedge \text{ for each continuous t-norm } *.$$

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Definition

A fuzzy pseudometric (in the sense of Kramosil and Michalek) on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, +\infty)$ such that for all $x, y, z \in X,\ t, s > 0$:

(FM1) $M(x, y, 0) = 0$;

(FM2) $M(x, x, t) = 1$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(FM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;

If the fuzzy pseudometric $(M, *)$ also satisfies:

(FM2') $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$

then $(M, *)$ is said to be a fuzzy metric on X.

In this case, $(X, M, *)$ is said to be a fuzzy (pseudo)metric space.

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Every fuzzy (pseudo)metric $(M, *)$ on X generates a uniformity $\mathcal{U}_M$ on X which has as a base the family $\{U_n : n \in \mathbb{N}\}$ where

$$U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}.$$

Standard fuzzy pseudometric

Example

Let (X, d) be a pseudometric space. Let M_d be the fuzzy set on $X \times X \times [0, \infty)$ given by

$$M_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For every continuous t-norm $*$, $(M_d, *)$ is a fuzzy pseudometric on X which is called the *standard fuzzy pseudometric* induced by d. Furthermore, we notice that $\mathcal{U}_d = \mathcal{U}_{M_d}$ where $\mathcal{U}_d$ is the uniformity generated by d.

Definition

A base of fuzzy pseudometrics on a nonempty set X is a pair $(\mathcal{B}, *)$ where $*$ is a continuous t-norm and $\mathcal{B}$ is family of fuzzy pseudometrics on X with respect to the t-norm $*$ closed under finite infimum.
If no confusion arises, we will write $M \in \mathcal{B}$ whenever $(M, *) \in \mathcal{B}$.

Notation

If $(M,*)$ is a fuzzy (pseudo)metric on X we will denote by M_t the function on $X \times X$ given by $M_t(x,y) = M(x,y,t)$ for all $t > 0$.

Definition

Let $(\mathcal{B},*)$ be a base of fuzzy pseudometrics on a nonempty set X. We define:

- $\langle \mathcal{B} \rangle = \{(N,*) \in \text{FMet}(*):$ for all $t > 0$ there exists $M \in \mathcal{B}$ and $s > 0$ such that $M_s \leq N_t\}$.
- $\widetilde{\mathcal{B}} = \{(N,*) \in \text{FMet}(*):$ for all $\varepsilon \in]0,1]$ and $t > 0$ there exist $s > 0, M \in \mathcal{B}$ such that $M_s - \varepsilon \leq N_t\}$.
- $\widehat{\mathcal{B}} = \{(N,*) \in \text{FMet}(*):$ for all $\varepsilon \in]0,1]$ and $t > 0$ there exist $\delta \in]0,1], s > 0, M \in \mathcal{B}$ such that $M(x,y,s) > 1-\delta$ implies $N(x,y,t) > 1-\varepsilon\}$.

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If $(M, *)$ is a fuzzy (pseudo)metric on X we will denote by M_t the function on $X \times X$ given by $M_t(x, y) = M(x, y, t)$ for all $t > 0$.

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Lemma

Let $(\mathcal{B}, *)$ *be a base of fuzzy pseudometrics on a nonempty set* X*. Then:*

$$\mathcal{B} \subseteq \langle \mathcal{B} \rangle \subseteq \widetilde{\mathcal{B}} \subseteq \widehat{\mathcal{B}}.$$

Furthermore, all these operators are idempotent.

Definition (Gutiérrez García, Romaguera and Sanchis, 2010)

Let X be a nonempty set and let $*$ be a continuous t-norm. A fuzzy uniform structure for $*$ is base of fuzzy pseudometrics $(\mathcal{M}, *)$ on X such that:

$$\widehat{\mathcal{M}} = \mathcal{M}.$$

A fuzzy uniform space is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a fuzzy uniform structure on X.

Definition (Gutiérrez García, Romaguera and Sanchis, 2010)

Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two fuzzy uniform spaces. A mapping $f : X \to Y$ is said to be uniformly continuous if for each $N \in \mathcal{N}$, $\varepsilon \in (0,1)$ and $t > 0$ there exist $M \in \mathcal{M}$, $\delta \in (0,1)$ and $s > 0$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - \delta$.

FUnif $\equiv$ category of fuzzy uniform spaces

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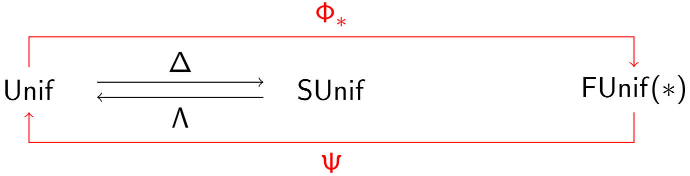
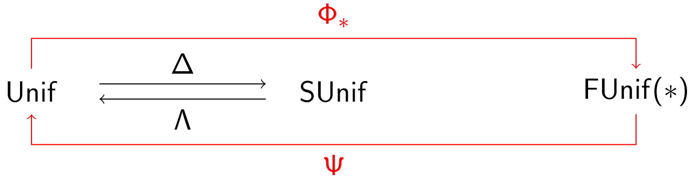
Φ_*
Unif
Δ
Λ
SUnif
FUnif$(*)$


Φ*
Δ
Unif
SUnif
FUnif(∗)
Λ
Ψ



- $\Phi_*(\mathcal{U}) = \langle \{M_d : d \text{ belongs to the uniform structure of } \mathcal{U}\} \rangle$;
- $\Psi(\mathcal{M}) = \bigvee_{M \in \mathcal{M}} \mathcal{U}_M$.

Definition (Höhle 78, Katsaras 79)

A probabilistic uniformity on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a continuous t-norm and $\mathcal{U}$ is a prefilter on $X \times X$ such that:

(FU1) $U(x,x) = 1$ for all $U \in \mathcal{U}$;

(FU2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where $U^{-1}(x,y) = U(y,x)$;

(FU3) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$V^2 \leq U$$

where $V^2(x,y) = \sup_{z \in X} V(x,z) * V(z,y)$;

In this case we say that $(X, \mathcal{U}, *)$ is a probabilistic uniform space.

Definition (Lowen 81, Höhle 82)

A Lowen $*$-uniformity on a nonempty set X is a saturated probabilistic uniformity $(\mathcal{U}, *)$ on X, i. e. a probabilistic uniformity $(\mathcal{U}, *)$ such that

$$\sup_{\varepsilon \in]0,1]} (U_\varepsilon - \varepsilon) \in \mathcal{U} \text{ whenever } \{U_\varepsilon : \varepsilon \in]0,1]\} \subseteq \mathcal{U}.$$

Definition

A function $f : (X, \mathcal{U}, *) \to (Y, \mathcal{V}, \star)$ between two probabilistic uniform spaces is said to be fuzzy uniformly continuous if for every $V \in \mathcal{V}$ we can find $U \in \mathcal{U}$ such that

$$U(x, y) \leq V(f(x), f(y)) \text{ for all } x, y \in X.$$

PUnif $\equiv$ category of probabilistic uniform spaces
LUnif $\equiv$ category of spaces endowed with a Lowen uniformity

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Lowen functors

$\omega_* : \mathsf{Unif} \to \mathsf{LUnif}(*)$

$$\omega_*(\mathcal{U}) = \{F \in I^{X \times X} : F^{-1}(]\varepsilon, 1]) \in \mathcal{U} \text{ for all } \varepsilon \in [0, 1[\}$$

$\iota : \mathsf{LUnif} \to \mathsf{Unif}$

$$\iota(\mathfrak{U}) = \{U^{-1}(]\varepsilon, 1]) : U \in \mathfrak{U}, \varepsilon \in [0, 1[\}$$

- ω_* and ι are adjoint functors;
- $\iota(\omega_*(\mathcal{U})) = \mathcal{U}$;
- $\mathfrak{U} \subseteq \omega_*(\iota(\mathfrak{U}))$.

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Probabilistic uniform structures

Definition

Let X be a nonempty set and let $*$ be a continuous t-norm. A probabilistic $*$-uniform structure (resp. Lowen $*$-uniform structure) on X is base of fuzzy pseudometrics $(\mathcal{M}, *)$ on X such that

$$\langle \mathcal{M} \rangle = \mathcal{M}$$

$$(\text{resp. } \widetilde{\mathcal{M}} = \mathcal{M}).$$

A space with a probabilistic $*$-uniform structure (resp. Lowen $*$-uniform structure) is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a probabilistic $*$-uniform structure (resp. Lowen $*$-uniform structure) on X (the t-norm $*$ will be omitted if no confusion arises).

Definition

Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two spaces endowed with two probabilistic uniform structures. A mapping $f : X \to Y$ is said to be fuzzy uniformly continuous if for every $(N, \star) \in \mathcal{N}$ and $t > 0$ there exist $(M, *) \in \mathcal{M}$ and $s > 0$ such that $M(x, y, s) \leq N(f(x), f(y), t)$ for all $x, y \in X$.

PSUnif $\equiv$ category of spaces endowed with a probabilistic uniform structure

LSUnif $\equiv$ category of spaces endowed with a Lowen uniform structure

Definition

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$M(x, y, s) \leq N(f(x), f(y), t)$ for all $x, y \in X$.

PSUnif $\equiv$ category of spaces endowed with a probabilistic uniform structure
LSUnif $\equiv$ category of spaces endowed with a Lowen uniform structure

Proposition

LSUnif$(*)$ *is a coreflective subcategory of* PSUnif$(*)$ *whose coreflector is the functor* $\mathcal{S}_s : \mathrm{PSUnif}(*) \to \mathrm{LSUnif}(*)$ *given by* $\mathcal{S}_s((X, \mathcal{M}, *)) = (X, \widetilde{\mathcal{M}}, *)$ *and leaving morphisms unchanged.*

Proposition

FUnif$(*)$ *is a coreflective subcategory of* LSUnif$(*)$ *whose coreflector is the functor* $\iota_s : \mathrm{LSUnif}(*) \to \mathrm{FUnif}(*)$ *given by* $\iota_s((X, \mathcal{M}, *)) = (X, \widehat{\mathcal{M}}, *)$ *and leaving morphisms unchanged.*

Proposition

$\mathsf{LSUnif}(*)$ *is a coreflective subcategory of* $\mathsf{PSUnif}(*)$ *whose coreflector is the functor* $\mathcal{S}_s : \mathsf{PSUnif}(*) \to \mathsf{LSUnif}(*)$ *given by* $\mathcal{S}_s((X, \mathcal{M}, *)) = (X, \widetilde{\mathcal{M}}, *)$ *and leaving morphisms unchanged.*

Proposition

$\mathsf{FUnif}(*)$ *is a coreflective subcategory of* $\mathsf{LSUnif}(*)$ *whose coreflector is the functor* $\iota_s : \mathsf{LSUnif}(*) \to \mathsf{FUnif}(*)$ *given by* $\iota_s((X, \mathcal{M}, *)) = (X, \widehat{\mathcal{M}}, *)$ *and leaving morphisms unchanged.*

Proposition

Let us consider the map $\mathfrak{S} : \mathsf{PUnif} \to \mathsf{PSUnif}$ *given by*

$$\mathfrak{S}((X, \mathcal{U}, *)) = (X, \mathfrak{s}(\mathcal{U}), *) = (X, \mathcal{M}_{\mathcal{U}}, *)$$

where $(\mathfrak{s}(\mathcal{U}), *) = (\mathcal{M}_{\mathcal{U}}, *)$ *is the probabilistic uniform structure of all fuzzy pseudometrics* $(M, *)$ *on* X *such that* $M_t \in \mathcal{U}$ *for all* $t > 0$ *and*

$$\mathfrak{S}(f) = f$$

for every morphism f *in* PUnif. *Then* $\mathfrak{S}$ *is a covariant fully faithful functor.*

Proposition

Let us consider the map $\Upsilon : \mathsf{PSUnif} \to \mathsf{PUnif}$ *given by*

$$\Upsilon((X, \mathcal{M}, *)) = (X, \upsilon(\mathcal{M}), *) = (X, \mathcal{U}_{\mathcal{M}}, *)$$

where $(\mathcal{U}_{\mathcal{M}}, *)$ *is the probabilistic uniformity which has as base the family* $\{M_t : t > 0, (M, *) \in \mathcal{M}\}$ *and*

$$\Upsilon(f) = f$$

for every morphism f *in* PSUnif. *Then* Υ *is a fully faithful covariant functor.*

Theorem

$\mathfrak{S} \circ \Upsilon = 1_{\mathsf{PSUnif}}$ and $\Upsilon \circ \mathfrak{S} = 1_{\mathsf{PUnif}}$ so the categories PSUnif *and* PUnif *are isomorphic.*

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Theorem

The following diagram commutes:

$$
\begin{array}{ccc}
\mathsf{Unif} & \underset{\Psi}{\overset{\Phi_*}{\rightleftarrows}} & \mathsf{FUnif}(*) \\
\downarrow{\scriptstyle \omega_*} & & \downarrow{\scriptstyle i} \\
\mathsf{LUnif}(*) & \underset{\mathfrak{S}}{\overset{\Upsilon}{\leftrightarrows}} & \mathsf{LSUnif}(*) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
\mathsf{PUnif}(*) & \underset{\mathfrak{S}}{\overset{\Upsilon}{\leftrightarrows}} & \mathsf{PSUnif}(*)
\end{array}
$$

where i denotes the inclusion functor.

Theorem

The following diagram commutes:

$$\begin{array}{ccc}
\mathsf{PUnif}(*) & \underset{\Upsilon}{\overset{\mathfrak{S}}{\rightleftarrows}} & \mathsf{PSUnif}(*) \\
\Big\downarrow \mathcal{S} & & \Big\downarrow \mathcal{S}_s \\
\mathsf{LUnif}(*) & \underset{\Upsilon}{\overset{\mathfrak{S}}{\rightleftarrows}} & \mathsf{LSUnif}(*) \\
\Big\downarrow \iota & & \Big\downarrow \iota_s \\
\mathsf{Unif} & \underset{\Psi}{\overset{\Phi_*}{\rightleftarrows}} & \mathsf{FUnif}(*)
\end{array}$$