Recovering a compact Hausdorff space X from the Compatibility Ordering on $C(X)$

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(joint with Tomasz Kania)

Prague, 29 July 2016
TOPOSYM

Basic Definitions

- X and Y are compact Hausdorff topological spaces;
- $C(X)$ is the set of all continuous functions $f : X \to \mathbb{R}$.

Definition (Compatibility Ordering)

Let $f, g \in C(X)$. We write

$$f \preceq g \overset{def}{\Longleftrightarrow} f(x) = g(x) \text{ for each } x \in \operatorname{supp} f.$$

$T : (C(X), \preceq) \to (C(Y), \preceq)$ is a compatibility morphism if

$$\forall f, g \in C(X) : \quad f \preceq g \implies Tf \preceq Tg.$$

T is a compatibility isomorphism if it is bijective and $\Longleftrightarrow$.

- $\preceq$ is a partial order on $C(X)$;
- zero function is the least element (i.e. $\forall f : 0 \preceq f$);
- if $T : C(X) \to C(Y)$ is a c. isomorphism, then $T(0) = 0$.

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Main Theorem

Theorem (T.Kania & M.R.)

Let X and Y be compact Hausdorff spaces, and let there exist a compatibility isomorphism $T : C(X) \to C(Y)$. Then X and Y are homeomorphic.

Sketch of proof: T behaves nicely w.r.t. supports.
More precisely: Given $f \in C(X)$, set

$$\sigma(f) = \operatorname{Int} \operatorname{supp}(f)$$

and define $\tau : \{\sigma(f) : f \in C(X)\} \to \{\sigma(g) : g \in C(Y)\}$ as

$$\tau(\sigma(f)) := \sigma(Tf).$$

Then τ is well-defined. And it is a $\subseteq$-isomorphism between bases of the topologies on X and Y.
Use this to define a homeomorphism.

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X and Y compact Hausdorff spaces, $T : C(X) \to C(Y)$ bijection.

Theorem (Gelfand–Kolmogorov, 1939)

T is a ring isomorphism $\implies X \sim Y$.

Theorem (Milgram, 1949)

T is multiplicative $\implies X \sim Y$.

Proof. We want: multiplicative bij. $\implies$ compatibility iso.
Then we apply the Main Theorem to conclude $X \sim Y$.
To that end, we need to observe:

- $f, g \in C(X)$. Then $f \preceq g \iff fg = f^2$.
- T multiplicative bijection $\implies T^{-1}$ multiplicative.

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X and Y compact Hausdorff spaces, $T : C(X) \to C(Y)$ bijection.

Theorem (Kaplansky, 1947)

T is a lattice isomorphism $\Longrightarrow X \sim Y$.

Lattice isomorphism $\equiv$ for all $f, g \in C(X)$,
$T(\max\{f,g\}) = \max\{Tf, Tg\}$ and $T(\min\{f,g\}) = \min\{Tf, Tg\}$.

Proof: We need to observe:

- It is enough to consider $f, g \geq 0$.
- Then $f \preceq g \Longleftrightarrow f \leq g \quad \& \quad \max\{g-f, f\} \geq g$.
- Lattice isomorphism $\equiv$ pointwise-order isomorphism.

It follows that any lattice isomorphism is compatibility isomorphism.
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Theorem (Jarosz, 1990)

T is linear, disjointness preserving $\Longrightarrow X \sim Y$.

Disjointness preserving $\equiv \forall f, g \in C(X) : f \cdot g = 0 \Longrightarrow Tf \cdot Tg = 0$.

Proof: Literature $\rightsquigarrow T^{-1}$ disj. preserving.
We show that T preserves $\preceq$; the proof for T^{-1} is the same.
Fix $f, g \in C(X)$ with $f \preceq g$. Then $g - f$ and f are non-overlapping.
By the assumption on T, $T(g - f)$ and f are also disj. supp.
Hence $Tg = T(g - f) + Tf \succeq Tf$.
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Remark

We do not use: Maximal ideals in $C(X)$ are kernels of Diracs.
This fact is crucial in the original proofs of these theorems.

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Two partial results & Main Thm again

X and Y compact Hausdorff spaces, $T : C(X) \to C(Y)$ bijection.

Theorem (T.K. & M.R.)

Let X be sequentially compact and let all of its components be nowhere dense. Then every compatibility isomorphism $T: C(X) \to C(Y)$ is **norm-continuous**.

Theorem (T.K. & M.R.)

If X contains a locally connected open subset, then there exist c. isomorphisms which are **not continuous**.

Observation leading to a proof

X connected and $\forall x : f(x) \neq 0 \implies$ f minimal.

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X and Y compact Hausdorff spaces, $T : C(X) \to C(Y)$ bijection.

Theorem (T.K. & M.R.)

Let X be sequentially compact and let all of its components be nowhere dense. Then every compatibility isomorphism $T: C(X) \to C(Y)$ is **norm-continuous**.

Theorem (T.K. & M.R.)

If X contains a locally connected open subset, then there exist c. isomorphisms which are **not continuous**.

Theorem (Main Theorem)

T is a compatibility isomorphism $\implies X \sim Y$.

Thank you for your attention.