Extremally disconnected topological groups

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Contents

The problem

Problem (Arhangel'skii, 1967)

Is there a nondiscrete extremally disconnected topological group?

Definition (Stone, 1937)

A topological space is called *extremally disconnected* (or *ED* for short) if it is regular and the closure of every open set is open, or equivalently, the closures of any two disjoint open sets are disjoint.

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Elementary facts about ED spaces

- Every ED space is zero-dimensional.
- Every open (or dense) subspace of an ED space is also an ED space.
- Extremal disconnectedness is preserved under open continuous surjection maps.
- Every discrete space is ED, but the converse is not true (*e.g.*, $\beta\omega$).
- Every sequence in an ED space is trivial. In particular, every metrizable ED space is discrete.

Extremal disconnectedness can be considered as a **non-trivial generalization of discreteness**. This notion has been studied by many authors for several years.

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Consistent examples

Partial positive solutions

For each one of the following assumptions, there is an example answering Arhangel'skii's question:

- (Sirota, 1969/Louveau, 1972) There is a selective ultrafilter on ω.
- (Malykhin,1975) $\mathfrak{p} = \mathfrak{c}$.

These group topologies are on the countable Boolean group $([\omega]^{<\omega}, \Delta)$. In fact, Arhangel'skii's question can be reduced to the Boolean case.

Theorem (Malykhin, 1975)

Any ED topological group must contain an open (and therefore closed) Boolean subgroup (i.e., a subgroup consisting of elements of order 2).

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The classical consistent examples

Given a filter $\mathcal{F}$ on ω, $\mathcal{F}^{<\omega} = \{[F]^{<\omega} \colon F \in \mathcal{F}\}$ induces a group topology $\tau_{\mathcal{F}}$ on $\mathbb{B}(\omega)$ by declaring $\mathcal{F}^{<\omega}$ to be the filter of neighbourhoods of the $\emptyset$.

Theorem (Louveau, 1972)

The group $(\mathbb{B}(\omega), \tau_{\mathcal{F}})$ is ED if and only if $\mathcal{F}$ is a selective ultrafilter.

The same works on a measurable cardinal and yet another example can be obtained from Matet forcing with an ordered-union ultrafilter on $\mathbb{B}^{+}(\omega) := [\omega]^{<\omega} \setminus \{\emptyset\}$.

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Not adding Cohen reals (this part is joint work with M. Hrušák)

In the study of forcing notions is particularly important when some kind of forcing notions adds or does not add Cohen reals.

Proposition

Let X be an ED space. Then $RO(X)$ does not add Cohen reals if and only if for every continuous function $f : X \to 2^\omega$ there exists a non-empty open set U such that $f''U \in \text{nwd}(2^\omega)$.

- If X is a countable space then $RO(X)$ is a σ-centered forcing notion.

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nwd-ultrafilters

Theorem (Błaszczyk-Shelah, 2001)

The following are equivalent.

- *There is a* nwd-*ultrafilter on* ω.
- *There is a non-trivial* σ*-centered forcing notion which does not add Cohen reals.*

Definition (Baumgartner, 1995)

An ultrafilter p on ω is *nowhere dense* (nwd) if for any $f : \omega \to 2^\omega$ there is $A \in p$ such that $f''A \in \text{nwd}(2^\omega)$.

- (Baumgartner, 1995) Every P-point is a nwd-ultrafilter.

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It is consistent with ZFC *that there is no* nwd-*ultrafilter on* ω.

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$RO(\mathbb{G})$ and a conjecture

The classical consistent examples for Arhangel'skii's question satisfy the following property.

- For every continuous function $f: \mathbb{G} \to 2^\omega$ there exists a non-empty open set U such that $f''U \in \mathrm{nwd}(2^\omega)$.

Is this a simple accident?

Conjecture (Hrušák)

For every ED topological group $\mathbb{G}$ and for every continuous function $f: \mathbb{G} \to 2^\omega$ there is an non-empty open set U such that $f''U \in \mathrm{nwd}(2^\omega)$.

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More evidence

Theorem

Let $\mathbb{G}$ be an ED topological group. If $f : \mathbb{G} \to 2^\omega$ is a continuous homomorphism, then there is a non-empty open set U such that $f''U \in \mathrm{nwd}(2^\omega)$.

Proof.

WLOG f is a continuous monomorphism. For every $n \in \omega$ let $\sigma_n \in 2^{n+1}$ be such that $\sigma_n(i) = 1$ iff $i = n$. Put

$$U_0 = \bigsqcup_{n \text{ even}} f^{-1}[\sigma_n] \quad \text{and} \quad U_1 = \bigsqcup_{n \text{ odd}} f^{-1}[\sigma_n].$$

Then $\mathbb{G} \setminus \{e_{\mathbb{G}}\} = U_0 \sqcup U_1$. Since $\mathbb{G}$ is ED group, there exists $i \in 2$ and U an open neighbourhood of $e_{\mathbb{G}}$ such that $U \cdot U \subset U_i \cup \{e_{\mathbb{G}}\}$. It is easy to see that $f''U \in \mathrm{nwd}(2^\omega)$. □

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Question

Is it true the Hrušák's conjecture?

If Hrušák's conjecture is true, then the existence of a nondiscrete separable ED topological group implies the existence of a nwd-ultrafilter on ω and thus, the existence of a nondiscrete separable ED topological group will be independent of ZFC.

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Theorem (Sipacheva, 2015)

The existence of a countable ED Boolean topological group with many open subgroups (i.e., containing a family of open subgroups whose intersection has empty interior) implies the existence of a rapid ultrafilter on ω.

The ideas contained in the proof of this theorem allow isolate the following notion:

Definition

Let $\mathbb{G}$ be a Boolean topological group. A sequence $\{e_\beta : \beta < \theta\} \subset \mathbb{G}$ is called *algebraic free* if for all $\beta < \theta$

$$\overline{\mathrm{span}}\{e_\alpha : \alpha \leqslant \beta\} \cap \overline{\mathrm{span}}\{e_\alpha : \beta < \alpha < \theta\} = \{0_{\mathbb{G}}\}.$$

It is *nontrivial* if $\mathrm{span}\{e_\beta : \beta < \theta\}$ is a nondiscrete subgroup of $\mathbb{G}$.

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It is *nontrivial* if $\mathrm{span}\{e_\beta \colon \beta < \theta\}$ is a nondiscrete subgroup of $\mathbb{G}$.

Proposition

Let $\mathbb{G}$ be a countable Boolean topological group. Then

1. *$\mathbb{G}$ admits an infinite algebraic free sequence.*
2. *If $\mathbb{G}$ has many open subgroups, then $\mathbb{G}$ admits an algebraic free sequence which generates $\mathbb{G}$.*

Theorem

Let $\mathbb{G}$ be a nondiscrete ED Boolean topological group containing a countable nontrivial free sequence. Then there is a rapid ultrafilter on ω.

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- What about in the Miller model or Laver model?

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Thank you!