Embedding cartesian products in symmetric products

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Joint work with
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Structure of the talk

1. Basic definitions and notations
2. The case $\mathrm{Ram}(G) = \emptyset$
3. $G = T_m$ is a simple m-od
4. Some results

Embedding products in symmetric products

Some notation

Continuum

A continuum X is a nonempty, compact, connected metric space.

Symmetric products

$$F_n(X) = \{A \in 2^X : A \text{ contains at most } n \text{ points}\}$$

where $2^X = \{A \subset X : A \text{ is closed and nonempty}\}$.

2^X is endowed with the Hausdorff metric.

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Finite graphs

A finite graph is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points

Some notatios on finite graphs

If G is a finite graph we denote by

- $\deg(x)$ the degree of a a point $x \in G$,
- $\mathrm{Ram}(G)$ the set of ramification points of G, i.e points with $\deg(x) \geq 3$.

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Is there an embedding $X^n \hookrightarrow F_n(X)$?

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Embedding products in symmetric products

$\mathrm{Ram}(G)=\emptyset$

There are only two cases for wich $\mathrm{Ram}(G)=\emptyset$: $G\simeq[0,1]$ and $G\simeq S^1$.

Proposition

For each $n\in\mathbb{N}$ there is an embedding $[0,1]^n\hookrightarrow F_n([0,1])$.

$\mathrm{Ram}(G) = \emptyset$

Theorem

For each $n \geq 2$ there is no embedding $\mathbb{T}^n := (S^1)^n \hookrightarrow F_n(S^1)$.

- For $n = 2$, $F_2(S^1)$ is the Möbius strip.
- The case $n = 3$ is a result of E. Castañeda.
- $n \geq 4$: Koyama and Chinen showed that $F_n(S^1)$ contains no copy of any orientable n-dimensional topological manifold.

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Embedding products in symmetric products

The simple m-od

We denote by T_m the simple m-od.

Theorem (E. Castañeda, J. Sánchez)

For $m \geq 3$ there is no embedding $T_m^2 \hookrightarrow F_2(T_m)$.

The proof uses the fact that T_m^2 is homeomorphic to $\mathrm{Cone}(K_{m,m})$ and $F_2(T_m)$ is homeomorphic to $\mathrm{Cone}(Z)$, where Z is homeomorphic to the complete graph K_m with some arcs at the vertices:

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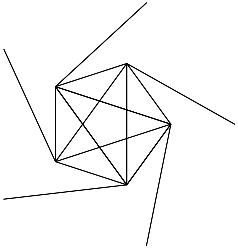

Theorem

There is an embedding $T_m^2 \hookrightarrow F_3(T_m)$

Proof: $F_3(T_m) \simeq \mathrm{Cone}(\mathcal{A})$, where

$$\mathcal{A} = \{C \in F_3(T_m) : C \cap \{x \in T_m : \deg(x) = 1\} \neq \emptyset\}.$$

$\mathcal{A}$ contains a copy of a torus with m transversal discs, say $T(m)$
If we get an embedding of $K_{m,m}$ in $T(m)$ we have done because we will get an embedding

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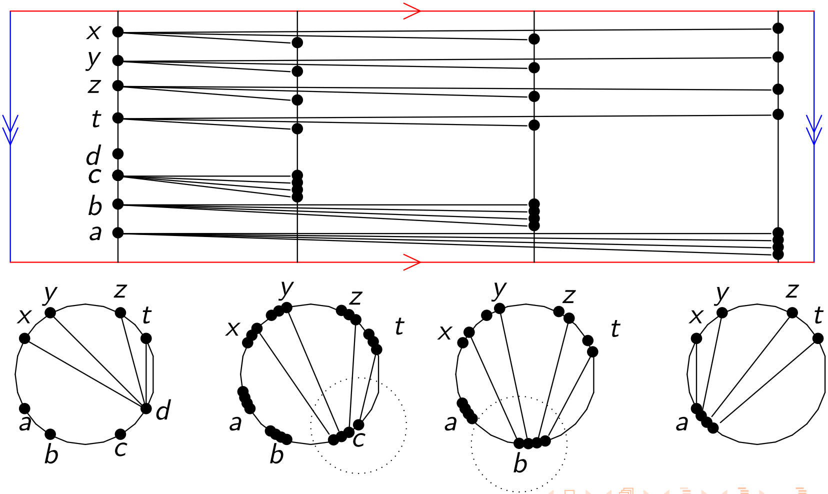

Embedding products in symmetric products

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There is no embedding $T_m^3 \hookrightarrow F_3(T_m)$ for $m \geq 3$.

The same ideas can be used to proof

Theorem

If G is a graph with $|\mathrm{Ram}(G)| \geq 2$ there is no embedding

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The key observation is about the type of neighbourhoods of $(p, q, r) \in G^3$, where $p, q, r \in \mathrm{Ram}(G)$: there is no embedding

$$T_m \times T_n \times T_s \hookrightarrow (F_2(T_k) \times [0,1]) \cup (T_j \times [0,1]^2)$$

if $k, j \leq \min\{m, n, s\}$

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Lemma

Let G be a finite graph, $p, q \in \mathrm{Ram}(G)$, $p \neq q$, with the property that $\deg(x) \leq \deg \min\{\deg(p), \deg(q)\}$ for all $x \in G$. Then for each embedding $h : G^3 \longrightarrow F_3(G)$ we have

$$|h(p, q, x) \cap \mathrm{Ram}(G)| \geq 2$$

This proof that in the case $\mathrm{Ram}(G) = 2$ there is no embedding.

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$$|h(p, q, r) \cap \mathrm{Ram}(G)| = 3$$

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We get a characterization of the arc as follows

Corollary

For a finite graph G, there is an embedding $G^3 \hookrightarrow F_3(G)$ if and only if G is homeomorphic to the arc $[0,1]$.

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E. Castañeda, *Symmetric products as cones and products*, Top. Proceedings 28 (2004) p.235-244

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Thank you