On lineability of classes of functions with various degrees of (dis)continuity

Krzysztof Płotka
University of Scranton

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We also define $\mathcal{L}_E(\mathcal{F})$, the **lineability of the family** $\mathcal{F}$ **over the field** E as

$$\mathcal{L}_E(\mathcal{F}) = \min\{\kappa\colon \mathcal{F} \text{ is not } \kappa\text{-lineable over } E\}.$$

In the case $E = \mathbb{R}$ we simply write $\mathcal{L}(\mathcal{F})$.

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Note: For any function $f\colon \mathbb{R} \to \mathbb{R}$, there exists a dense subset $S \subseteq \mathbb{R}$ such that the function $f|S$ is continuous (Blumberg, 1922). $\mathrm{AC} \subsetneq \mathrm{D}$. In addition, $\mathrm{AC} \cap \mathrm{SZ} \neq \emptyset$ and $\mathrm{D} \cap \mathrm{SZ} \neq \emptyset$ are both independent of ZFC.

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Theorem (Gámez-Merino, Seoane-Sepúlveda)

Let $\mathfrak{c} < \kappa \leq 2^{\mathfrak{c}}$. Then $\mathcal{L}(\mathrm{SZ}) > \kappa$ is equivalent to the existence of an additive group in $\mathrm{SZ} \cup \{0\}$ of cardinality κ.

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Open Problem

Is $\mathrm{AC} \cap \mathrm{SZ} \neq \emptyset$ and $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) < \mathcal{L}(\mathrm{SZ})$ consistent with ZFC?

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  Assuming CH, there is a function $h \in \mathrm{AC} \cap \mathrm{SZ}$ and an $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ which is h-negligible with respect to AC.

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We construct a sequence of partial functions h_α $(\alpha<\mathfrak{c})$ such that:

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