Weak network and a weaker covering property for the basis problem

Yinhe Peng
joint work with Stevo Todorcevic

University of Toronto

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To answer this question we are willing to use standard forcing axioms (MA, PFA,...), and/or restrict ourselves to some appropriate subclass of well-behaved spaces.

The real line and the Sorgenfrey line

Theorem (Baumgartner 1973)

PFA implies that every set of reals of cardinality $\aleph_1$ *embeds homomorphically into any uncountable regular space of countable network and that*

every subset of the Sorgenfrey line $(\mathbb{R}, \rightarrow)$ *of cardinality* $\aleph_1$ *embeds homomorphically into any uncountable subspace of* $(\mathbb{R}, \rightarrow)$*.*

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Note that neither an uncountable discrete space nor an uncountable subspace of the Sorgenfrey line has a countable network.

So even in the class of first countable spaces the list $\mathcal{B}$ must have at least three elements.

S and L spaces

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An L space is a regular hereditarily Lindelöf (HL) space which is not separable.

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So under PFA, an uncountable regular space either contains an uncountable discrete space or is HL.

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Theorem (Szentmiklossy, 1980)

MA_{ω_1} implies that there are no first countable L spaces.

One class

A topological space X is cometrizable if it has a weaker metrizable topology and a neighbourhood assignment consisting of closed sets in this weaker topology.

Example: The Sorgenfrey line is a cometrizable space.

Theorem (Gruenhage 1987)

Assume PFA. A cometrizable space has a countable network if it contains no uncountable discrete subspace nor an uncountable subspace of the Sorgenfrey line.

Another class

Definition

$\mathcal{C} = \{X_\alpha : \alpha < \kappa\} \subset P(X)$ *is a weak network if there is a base such that for every open set* O *in the base,* $O \setminus \cup\{X_\alpha : X_\alpha \subset O\}$ *is at most countable.*

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Obviously, separable metric space, Sorgenfrey subset and discrete space all have countable weak networks.

Theorem (PFA)

If a regular space X has a countable weak network, then either X has a countable network or X contains an uncountable subset which is either discrete or Sorgenfrey.

Weak network

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If X is a regular space and is weaker than the Sorgenfrey topology when mod $[X]^{\leq\omega}$, then X contains an uncountable metrizable or Sorgenfrey subset.

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If X is a regular space and is weaker than the Sorgenfrey topology when mod $[X]^{\leq\omega}$, then X contains an uncountable metrizable or Sorgenfrey subset.

Corollary (Gruenhage; PFA)

If there is a regular space contains none of the 3 spaces, there is a sub-metrizable one.

Applications to other problem

Theorem (PFA)

If X is a regular HL space with a countable weak network, then X admits a 2-to-1 continuous map to a metric space.

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A similar question in perfect normal compact spaces has drawn people's attention for a long time.

Question (Fremlin)

Is it consistent that every perfectly normal compact space admits a 2-to-1 continuous map to a metric space?

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Question

Is it consistent that every perfectly normal locally connected compact space is metrizable?

Question

If X and Y are compact and $X \times Y$ is perfectly normal, must one of X and Y be metrizable?

First countability

Theorem (PFA)

If X is a first countable regular HL space of size $\aleph_1$ with a countable weak network, then there is a partition $X = \bigcup_{n<\omega} X_n$ such that each X_n is either metrizable or Sorgenfrey.

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Corollary (PFA)

If f is a continuous 1-1 map from a Sorgenfrey subset of size $\aleph_1$ to a first countable regular space, then it is a countable union of sub-maps such that each sub-map is either a homeomorphism or a map from Sorgenfrey to metrizable space.

Inner topology

Definition

For a topological space (X, τ) *and a collection* $\mathcal{C} \subset P(X)$*, the inner topology* $(X, \tau^{I,\mathcal{C}})$ *induced by* $\mathcal{C}$ *is the topology with base* $\{\{x\} \cup O^{I,\mathcal{C}} : x \in O, O \text{ is open}\}$ *where* $O^{I,\mathcal{C}} = \cup\{C \in \mathcal{C} : C \subset O\}$.

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If (X, τ) is regular and $(X, \tau^{I,\mathcal{C}})$ is HL for some countable $\mathcal{C}$, then (X, τ) either has a countable network or contains an uncountable Sorgenfrey subset.

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Proposition (PFA)

If X *is first countable, regular and contains no uncountable separable metrizable or Sorgenfrey subset, then for any countable collection* $\mathcal{C}$, $(X, \tau^{I,\mathcal{C}})$ *is a countable union of discrete subsets.*

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HL of inner topology is preserved under continuous image and perfect preimage for sub-metrizable spaces.

Outer "topology"

Definition

For a topological space (X, τ) *and a collection* $\mathcal{C} \subset P(X)$, *the outer "topology"* $(X, \tau^{O,\mathcal{C}})$ *induced by* $\mathcal{C}$ *is the collection* $\{O^{O,\mathcal{C}} : O$ *is open*$\}$ *where* $O^{O,\mathcal{C}} = \cap\{C \in \mathcal{C} : O \subset C\}$.

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Suppose X *is a regular, HL space. Any outer topology induced by a countable collection either has a countable network or contains an uncountable Sorgenfrey subset.*

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Suppose X is a regular, HL space. Any outer topology induced by a countable collection either has a countable network or contains an uncountable Sorgenfrey subset.

If the outer topology guesses almost correctly, then the original topology will either have a countable network or contain an uncountable Sorgenfrey subset.

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Example. Cometrizable spaces.

Outer "topology" to covering property

Proposition (PFA)

Suppose X is a first countable regular, HL space, $\mathcal{C}$ is countable such that $(X, \tau^{O,\mathcal{C}})$ is metrizable and $(X, \langle \{x\} \cup (u^{O,\mathcal{C}}_{x,n} \setminus u_{x,n}) : x \in X \rangle)$ contains no uncountable HL subset for all n. Then X contains an uncountable metrizable subset.

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$[x, \infty) \cap Y' \subset u_{x,n}$ *for all* $x \in Y'$.

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Fact (MA_{ω_1})

Suppose that X is a first countable space with covering property (**): for any $m, n < \omega$, for any $\{a_\alpha \in X^n : \alpha < \omega_1\}$, there are $\alpha \neq \beta$ such that for any $i < n$, $a_\alpha(i) \in u_{a_\beta(i),m}$ and $a_\beta(i) \in u_{a_\alpha(i),m}$. Then X contains a metrizable subspace.

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Question

Is it consistent that X has an uncountable metrizable subspace if X^ω is HL?

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Maybe a weaker covering property for basis problem should involve real ordering.

Question (PFA)

Must X has a metrizable or Sorgenfrey subspace if X is first countable, regular and for all $m, n < \omega$, for every $\{a_\alpha \in X^n : \alpha < \omega_1\}$, there are $\alpha \neq \beta$ such that for any $i < n$, $\max\{a_\alpha(i), a_\beta(i)\} \in u_{\min\{a_\alpha(i), a_\beta(i)\}, m}$?

A weaker covering property

Definition

A first countable space X with a real ordering $<$ has property $(*)$ if

for any $n < \omega$, for any $(m_0, \ldots, m_{n-1}) \in \omega^n$, for any $\{a_\alpha, b_\alpha \in X^n : \alpha < \omega_1\}$ such that $b_\alpha(i) \in u_{a_\alpha(i), m_i} \cap (a_\alpha(i), \infty)$ whenever $\alpha < \omega_1, i < n$, there are $\alpha \neq \beta$ such that for any $i < n$, $b_\alpha(i) \in u_{a_\beta(i), m_i}$ and $b_\beta(i) \in u_{a_\alpha(i), m_i}$.

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Theorem (PFA)

Assume that X is a first countable regular space with property $(*)$ and X has no uncountable left sub-Sorgenfrey subspace. Then X contains an uncountable metrizable or Sorgenfrey subspace.

The role of ordering

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We will still succeed if we want a HL regular one. But first countable?

Thank you!