Crowded pseudocompact Tychonoff spaces of cellularity at most the continuum are resolvable

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Definitions

A crowded space is resolvable if it contains two disjoint dense sets and a space is pseudocompact if every continuous function with real values is bounded.

These two notions were introduced by Hewitt in the 1940's:

- E. Hewitt, *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 309–333.
- E. Hewitt, *Rings of real-valued continuous functions, I*, Trans. Amer. Math. Soc. (1948), 45–99.

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Is there a Hausdorff crowded space X such that every function $f : X \to \mathbb{R}$ is continuous at some point ?

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and the next observation of Malykhin:

the answer is yes iff there exists a Hausdorff Baire irresolvable space

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In the same paper they ask:

Is every Tychonoff crowded pseudocompact space resolvable?

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This is a natural question because every countable compact space is pseudocompact and every Tychonoff pseudocompact space is Baire

The first partial solution was given by van Mill who proved that every Tychonoff pseudocompact *ccc* space is $\mathfrak{c}$-resolvable

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Casarrubias-Segura, Hernández-Hernández and Tamariz-Mascarúa proved with MA that every T_2 crowded Baire *ccc* space is ω-resolvable

Dorantes-Aldama showed, assuming that $\mathfrak{c}$ is less than the first weakly inaccessible cardinal, that every T_2 crowded *ccc* Baire space is resolvable and the existence of a crowded *ccc* Baire space is equivalent to the existence of a measurable cardinal

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The first step is standard

Define a tree of nonempty open sets of X such that:

1. $\mathcal{U}_0 = \{X\}$,
2. $\mathcal{U}_\alpha$ is cellular for every $\alpha < \kappa$, and
3. $\{U \in \mathcal{U}_{\alpha+1} : U \subseteq V\}$ and $\{cl(U) : U \in \mathcal{U}_{\alpha+1}$ and $U \subseteq V\}$ are infinite maximal cellular families with respect to V for every $\alpha < \kappa$ and $V \in \mathcal{U}_\alpha$,
4. if $\{U_\xi : \xi < \alpha\}$ is a chain, then $cl(U_{\xi+1}) \subseteq U_\xi$.

κ is the ordinal number where the construction stops

Define

1. $F_\alpha = [\bigcap_{\xi<\alpha}(\bigcup \mathcal{U}_\xi)] \setminus (\bigcup \mathcal{U}_\alpha)$ for every $\alpha < \kappa$ and
   $F_\kappa = \bigcap_{\xi<\kappa}(\bigcup \mathcal{U}_\xi)$
2. $\Lambda = \{\alpha \leq \kappa : cof(\alpha) = \omega\}$

Observe that

$\kappa \leq c(X)^+$, and

if X is pseudocompact then $F_\alpha \neq \emptyset$ for each $\alpha \in \Lambda$

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By $c(X) = \omega$

Each open set of the tree is related to a function $f : X \to [0, 1]$, so for each $\alpha < \omega_1$ it is defined a function from X to a countable power of $[0, 1]$

The goal

By using the metrizability of each step under ω_1, prove that there is a π-net of cardinality $\mathfrak{c}$ conformed by Cantor sets.

van Mill observes that every open set is eventually divided by the tree

Cantor sets appears after ω divisions of an open set because of the pseudocompactness

such sets live in the images of the F_α with $\alpha \in \Lambda$

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When $c(X) > \omega$

The van Mill's way could lead us through uncountable powers and then we may lose metrizability. Fortunately some facts can be translated and used:

1. if $U \subseteq \bigcup_{\xi \leq \alpha} F_\xi$ for some $\alpha < \kappa$, then U is divided by the tree before α
2. by pseudocompactness, if $\alpha \in \Lambda$ and $\{U_\xi \in \mathcal{U}_\xi : \xi < \alpha\}$ is a chain, then $\left(\bigcap_{\xi < \alpha} U_\xi\right) \cap F_\alpha \neq \emptyset$

Definition

For each $n \in \mathbb{N}$, let

$$\Gamma_n = \left\{ f : \{0,1\}^n \to \bigcup_{\xi < \kappa} \mathcal{U}_\xi : \ f \text{ is injective and } f\left[\{0,1\}^n\right] \text{ is cellular} \right\}$$

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Definition

1. Given n, m, $f \in \Gamma_n$ and $g \in \Gamma_m$ we will say that g extends f if $n < m$ and the set $\{g(y) : g(y) \subseteq f(x)\}$ is the set $\{g(y) : y \text{ extends } x\}$ for every $x \in \{0,1\}^n$.
2. $f, g \in \Gamma$ are compatible if they are equal or if one of them extends the other.

Lemma

Let $\alpha \in \Lambda$ and $Y \subseteq X$. Suppose that there are a strictly increasing sequence of ordinals $(\alpha_n)_{n\in\mathbb{N}}$ cofinal in α and a chain of compatible functions $f_n : \{0,1\}^n \to \mathcal{U}_{\alpha_n}(Y)$. Then, for every sequence of ordinals $(\beta_n)_{n\in\mathbb{N}}$ cofinal in α there is a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and a chain of compatible functions $g_m : \{0,1\}^m \to \mathcal{U}_{\beta_{\phi(m)}}(Y)$.

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Given $\alpha \in \Lambda$

Fix a strictly increasing sequence of ordinals $(\alpha_n)_{n\in\mathbb{N}}$ and define $\mathbb{C}_\alpha(Y)$ as the set of all chains of compatible functions $\{f_n \in \Gamma_n : n \in \mathbb{N}\}$ so that there is a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ satisfying that $f_n[\{0,1\}^n] \subseteq \mathcal{U}_{\alpha_{\phi(n)}}(Y)$ for each $n \in \mathbb{N}$.

By the last lemma $\mathbb{C}_\alpha$ represent all the Cantor-like families in F_α

Observe that $|\mathbb{C}_\alpha(X)| \leq c(X)^\omega$

Given a chain $\{f_n : n \in \mathbb{N}\} \in \mathbb{C}_\alpha(Y)$, let $f_{\{f_n:n\in\mathbb{N}\}} : \{0,1\}^\omega \to P(F_\alpha)$ be the function defined by

$$f_{\{f_n:n\in\mathbb{N}\}}(x) = \Big(\big(\bigcap_{n\in\mathbb{N}} f_n(x|_n)\big) \setminus int\big(\bigcap_{n\in\mathbb{N}} f_n(x|_n)\big)\Big)$$

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Crucial Facts:

Fact I

By the definition of the tree, if $x, y \in \{0,1\}^\omega$, then

$$f_{\{f_n : n \in \mathbb{N}\}}(x) \cap f_{\{f_n : n \in \mathbb{N}\}}(y) = \emptyset \text{ iff } x \neq y$$

Fact II

Let X be pseudocompact and let O be an open set. If $\{f_n : n \in \mathbb{N}\} \in \mathbb{C}_\alpha(O)$, then $f_{\{f_n : n \in \mathbb{N}\}}(x) \cap cl(O) \neq \emptyset$ for every $x \in \{0,1\}^\omega$

Fact III

Suppose that X is pseudocompact and $c(X) \leq \mathfrak{c}$. If O is an open subset contained in $\bigcup_{\xi \in \Lambda_\alpha} F_\xi$ satisfying that $\mathcal{U}_\xi(V) \neq \emptyset$ for every nonempty open $V \subseteq O$ and every $\xi < \alpha$, then O is resolvable ($\mathfrak{c}$-resolvable)

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Sketch of the proof

1. Consider the dense subspace $\bigcup_{\alpha\in\Lambda} F_\alpha$
2. by transfinite induction on Λ with the order type will be showed that $int(\bigcup_{\xi\in\Lambda_\alpha} F_\xi)$ is resolvable when $int(\bigcup_{\xi\in\Lambda_\alpha} F_\xi) \neq \emptyset$
3. if $int(F_\alpha) \neq \emptyset$, then $int(F_\alpha)$ is resolvable
4. finally we must consider three cases
   Case I: The successor case follows from the inductive hypothesis
   Case II: $\alpha = sup\{\xi \in \Lambda_\alpha : \xi < \alpha\}$. If $int(\bigcup_{\xi<\alpha} F_\xi) \neq \emptyset$, then $\bigcup_{\eta<\alpha} int(\bigcup_{\xi\in\Lambda_\eta} F_\xi)$ is dense in $int(\bigcup_{\xi<\alpha} F_\xi)$. So procede by inductive hypothesis
   Case III: Let $A = \bigcup_{\eta<\alpha} int(\bigcup_{\xi\in\Lambda_\eta} F_\xi)$ and let Y be the family of open subsets V of $int(\bigcup_{\xi<\alpha} F_\xi) \neq \emptyset$ such that $\mathcal{U}_\xi(W) \neq \emptyset$ for every open $W \subseteq V$ and every $\xi < \lambda$. Then $\bigcup Y = int\Big(int(\bigcup_{\xi<\alpha} F_\xi) \setminus A\Big)$. So, A is resolvable by inductive hypothesis and the complement is dense by Fact III

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Thank you.