On quasi-uniform box products

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Introduction

The theory of uniform box product was conveyed for the first time in 2001 by Scott Williams during the ninth Prague International Topological Symposium (Toposym). He proved, for instance, that the box product has a compatible complete uniformity whenever each factor does and he showed that the box product of realcompact spaces is realcompact whenever that the index set has no subset of measurable cardinality.

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Some progress have been done on the concept of uniform box product. For instance, Bell defined a new product topology on the countably many copies of a uniform space, coarser than the box product but finer than the Tychonov product, which she called the uniform box product. Her new product was motivated by the idea of the supremum metric on the countably many copies of (compact) metric spaces.

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The infinite game of two-player on uniform spaces was defined by Bell and it is called the proximal game. Then a uniform space $(X, \mathcal{D})$ is called $\mathcal{D}$-proximal provided that the first player has winning strategy in a proximal game on $(X, \mathcal{D})$. Moreover, a space X is called proximal if the space X admits a compatible uniformity $\mathcal{D}$ for which X is $\mathcal{D}$-proximal. Therefore, it follows that any metric space is proximal with the natural uniformity induced by the metric but it is not true that any proximal space is metrizable.

Motivation

In 2014, during the 29th Summer Conference on Topology and its Applications in New York, we were interested by a question from Ralph Kopperman "is possible to generalize the concept of infinite game of two-player on generalized uniform spaces (for instance quasi-uniform spaces)?" To give an answer to the above question, it seems natural to study first the theory of uniform box product in the framework of quasi-uniform spaces because the theory of uniformities on a box product and related concepts are subsumed by the concept of a proximal uniform space due to Bell.

Product topology

It is well-known that the product topology on the Cartesian product $\prod_{i\in I} X_i$ of a family $(X_i, \mathcal{U}_i)_{i\in I}$ of quasi-uniform spaces as the topology induced by $\prod_{i\in I} \mathcal{U}_i$ the smallest quasi-uniformity on $\prod_{i\in I} X_i$ such that each projection map $\pi_i : \prod_{i\in I} X_i \to X_i$ whenever $i \in I$ is quasi-uniformly continuous.

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Furthermore, the set of the form $\{((x_i)_{i\in I}, (y_i)_{i\in I}) : (x_i, y_i) \in U_i\}$ whenever $U_i \in \mathcal{U}_i$ and $i \in I$ is a sub-base for the quasi-uniformity $\prod_{i\in I} \mathcal{U}_i$.

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Furthermore, the set of the form $\{((x_i)_{i\in I}, (y_i)_{i\in I}) : (x_i, y_i) \in U_i\}$ whenever $U_i \in \mathcal{U}_i$ and $i \in I$ is a sub-base for the quasi-uniformity $\prod_{i\in I} \mathcal{U}_i$.
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The quasi-uniformity $\prod_{i\in I}\mathcal{U}_i$ is called product quasi-uniformity on $\prod_{i\in I} X_i$.

Lemma

Let $(X, \mathcal{U})$ be a quasi-uniform space and $\prod_{i\in\mathbb{N}} X$ be product set of many copies of X. Then $\check{\mathcal{U}}_i = \{\check{U}_i : U \in \mathcal{U}$ and $i \in \mathbb{N}\}$ is a quasi-uniform base on $\prod_{i\in\mathbb{N}} X$ where

$$\check{U}_i = \left\{(x, y) \in \prod_{i\in\mathbb{N}} X \times \prod_{i\in\mathbb{N}} X : (x(i), y(i)) \in U\right\}$$

whenever $i \in \mathbb{N}$ and $U \in \mathcal{U}$.

Remark

Note that $\overline{G} \in \tau(\check{\mathcal{U}}_i)$ if and only if for any $x = (x_i)_{i \in \mathbb{N}} \in \overline{G}$ there exists $\check{U}_i \in \check{\mathcal{U}}_i$ such that $\check{U}_i(x) \subseteq \overline{G}$ whenever $U \in \mathcal{U}$ and $i \in \mathbb{N}$.

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Let $y = (y_i)_{i\in\mathbb{N}} \in \check{U}_i(x)$ whenever $U \in \mathcal{U}$ and $i \in \mathbb{N}$ if and only if $(x, y) \in \check{U}_i$ whenever $U \in \mathcal{U}$ and $i \in \mathbb{N}$. Thus for any $x, y \in \overline{G}$, we have $(x_i, y_i) \in U$ whenever $U \in \mathcal{U}$ and $i \in \mathbb{N}$. Hence $\overline{G}$ is open set with respect to the topology induced by product quasi-uniformity on $\prod_{i\in\mathbb{N}} X$.

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Observe that the uniformity $(\check{\mathcal{U}}_i)^s$ coincides with the uniformity base on $\prod_{i\in\mathbb{N}} X$ and the topology $\tau((\check{\mathcal{U}}_i)^s)$ induced by the uniformity $(\check{\mathcal{U}}_i)^s$ is the Tychonov topology on $\prod_{i\in\mathbb{N}} X$.

Lemma

Let $(X,\mathcal{U})$ *be a quasi-uniform space and* $\prod_{i\in\mathbb{N}} X$ *be product set of many copies of* X. *Then* $\widehat{\mathcal{U}}_\psi = \{\widehat{U}_\psi : \psi : \mathbb{N} \to \mathcal{U} \quad \text{is a function}\}$ *is a quasi-uniform base on* $\prod_{i\in\mathbb{N}} X$ *where*

$$\widehat{U}_\psi = \left\{(x,y) \in \prod_{i\in\mathbb{N}} X \times \prod_{i\in\mathbb{N}} X : \text{ whenever } n \in \mathbb{N}, (x(i), y(i)) \in \psi(i)\right\}$$

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If $(X,\mathcal{U})$ is a uniform space, then the quasi-uniformity $\widehat{\mathcal{U}}_\psi$ is exactly the uniformity $\check{\mathbb{D}}$ in Bell's sense.

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whenever $U \in \mathcal{U}$ *and* $\psi : \mathbb{N} \to \mathcal{U} \quad$ *is a function.*

If $(X,\mathcal{U})$ is a uniform space, then the quasi-uniformity $\widehat{\mathcal{U}}_\psi$ is exactly the uniformity $\check{\mathbb{D}}$ in Bell's sense. Therefore for any quasi-uniform space $(X,\mathcal{U})$, the topology $\tau(\widehat{\mathcal{U}}_\psi)^s)$ induced by the uniformity base $\widehat{\mathcal{U}}^s_\psi$ is the box topology on $\prod_{\alpha\in\mathbb{N}} X$.

Quasi-uniform box product

Theorem

Let $(X,\mathcal{U})$ be a quasi-uniform space and $\prod_{i\in\mathbb{N}} X$ be a the product set of many copies of X. Then $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$ is a quasi-uniform base on $\prod_{i\in\mathbb{N}} X$ where

$$\overline{U} = \left\{(x,y) \in \prod_{i\in\mathbb{N}} X \times \prod_{i\in\mathbb{N}} X : \ (x(i), y(i)) \in U \textit{ whenever } i \in \mathbb{N}\right\}$$

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whenever $U \in \mathcal{U}$.

Definition

Let $(X, \mathcal{U})$ be a quasi-uniform space. Then the quasi-uniformity $\overline{\mathcal{U}}$ is called constant quasi-uniformity on the product $\prod_{i\in\mathbb{N}} X$ and the pair $\left(\prod_{i\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is called quasi-uniform box product.

Some properties

For any $\left(\prod_{i\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ quasi-uniform box product of a quasi-uniform space $(X, \mathcal{U})$. We have

1. $\overline{U} \cap \overline{V} = \overline{U \cap V}$ whenever $U, V \in \mathcal{U}$.
2. $\overline{U}^{-1} = \overline{U^{-1}}$ whenever $U \in \mathcal{U}$.
3. $\overline{U^{-1}} \cap \overline{U} = \overline{U^s} = \overline{U}^s$ whenever $U \in \mathcal{U}$.

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Lemma

If $(X, \mathcal{U})$ is a quasi-uniform space and $\left(\prod_{i\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ its quasi-uniform box product, then

$$\overline{U}(x) = \prod_{i\in\mathbb{N}} (U(x(i)))$$

and

$$\overline{U}^{-1}(x) = \prod_{i\in\mathbb{N}} (U^{-1}(x(i))) = \overline{U^{-1}}(x)$$

whenever $U \in \mathcal{U}$ and $x \in \prod_{i\in\mathbb{N}} X$.

Examples

Example

Let V be an uncountable discrete space. If we equip the Fort space $X = V \cup \{\infty\}$ with the Pervin quasi-uniformity $\mathcal{P}$ with the subbase $\mathcal{S} = \{S_A : A \subseteq V \text{ finite}\}$ with $S_A = [A \times A] \cup [X \setminus A \times X]$.
Then $S_A^{-1} = [A \times A] \cup [X \times X \setminus A]$ with $A \in \tau$. Furthermore

$$S_A \cap S_A^{-1} = [A \times A] \cup [(X \setminus A) \times (X \setminus A)].$$

We observe that $S_A \cap S_A^{-1} \supseteq D_A$ the open subbase of the uniformity on X if A is finite subset of V. It turns out that if $a \in A$, then

$$S_A(a) = \{b \in X : (a, b) \in S_A\} = A.$$

If $a \notin A$, then $S_A(a) = \{b \in X : (a, b) \in S_A\} = \{b \in X\} = X$.
Similarly if $a \in A$, then

$$S_A^{-1}(a) = X$$

and if $a \in A$, then

$$S_A^{-1}(a) = X \setminus A.$$

If $x \in \prod_{i\in\mathbb{N}} X$, then

$$\overline{S_A}(x) = \left\{ y \in \prod_{i\in\mathbb{N}} X : (x(i), (y(i)) \in S_A \text{ whenever } i \in \mathbb{N} \right\}.$$

Hence

$$\overline{S_A}(x) = \prod_{x(i)\in A} A \quad \text{or} \quad \overline{S_A}(x) = \prod_{x(i)\notin A} X$$

Moreover, if $x \in \prod_{i\in\mathbb{N}} X$, then

$$\overline{S_A{}^{-1}}(x) = \left\{ y \in \prod_{i\in\mathbb{N}} X : (y(i), x(i)) \in S_A \text{ whenever } i \in \mathbb{N} \right\}$$

Hence

$$\overline{S_A{}^{-1}}(x) = \prod_{x(i)\in A} A \quad \text{or} \quad \overline{S_A{}^{-1}}(x) \prod_{x(i)\in X} (X \setminus A).$$

Therefore

$$\overline{S_A}(x) \cap \overline{S_A{}^{-1}}(x) = \prod_{x(i)\in A} A \quad \text{or} \quad \prod_{x(i)\notin A} (X \setminus A).$$

Example

If we equipped the Fort space $X = V \cup \{\infty\}$ with the quasi-uniformity $\mathcal{W}_F$ with the subbase $\{W_F : F \subseteq V \ F \text{ is finite}\}$ where $W_F = \triangle \cup [X \times (X \setminus F)]$. We have that

$$W_F \cap W_F^{-1} = (\triangle \cup [X \times (X \setminus F)]) \cap (\triangle \cup [(X \setminus F) \times X]) = \triangle \cup (X \setminus F)^2. \quad (1)$$

It follows that if $x \in F$, then

$$W_F(x) = \{x\} \cup (X \setminus F)$$

and

$$W_F^{-1}(x) = \{y \in X : (y, x) \in W_F\} = \{x\}.$$

If $x \notin F$, then we have

$$W_F(x) = X \setminus F$$

and

$$W_F^{-1}(x) = X.$$

Moreover, it follows that

$$W_F(x) \cap W_F^{-1}(x) = [\{x\} \cup (X \setminus F)] \cap \{x\} = \{x\}$$

whenever $x \in F$. Whenever $x \notin F$, we have

$$W_F(x) \cap W_F^{-1}(x) = X \setminus F \cap X = X \setminus F.$$

Let $\left(\prod_{i\in\mathbb{N}} X, \overline{\mathcal{W}_F}\right)$ be the quasi-uniform box product of X.
Then whenever $x \in \prod_{i\in\mathbb{N}} X$, we have

$$\overline{W_F}(x) = \prod_{x(i)\in F} \Big(\{x(i)\} \cup X \setminus F\Big) \quad \text{or} \quad \overline{W_F}(x) = \prod_{x(i)\notin F} (X \setminus F)$$

and

$$\overline{W_F^{-1}}(x) = \prod_{x(i)\in F} \{x\} \quad \text{or} \quad \overline{W_F^{-1}}(x) = \prod_{x(i)\notin F} X.$$

Properties of filter pairs

If $(\mathcal{F}, \mathcal{G})$ is a pair of filters on a quasi-uniform space $(X, \mathcal{U})$. Then $(\mathcal{F}, \mathcal{G})$ is called Cauchy filter pair provide that there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$ whenever $U \in \mathcal{U}$.

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Moreover, a quasi-uniform space is bicomplete (or pair complete) provided that whenever $(\mathcal{F},\mathcal{G})$ is Cauchy filter pair on $(X,\mathcal{U})$, there exists a point $x \in X$ such that $\mathcal{G}$ converges to x with resect to $\tau(\mathcal{U})$ and $\mathcal{F}$ converges to x with respect to $\tau(\mathcal{U}^{-1})$.

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Let $(X, \mathcal{U})$ be a quasi-uniform space and $\left(\prod_{i \in \mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $\mathcal{F}$ is a filter on X, then $\prod_{i \in \mathbb{N}} \mathcal{F}$ will denote the filter on $\prod_{i \in \mathbb{N}} X$ that is generated by the filter base consisting of sets $\prod_{i \in \mathbb{N}} F$ where $F \in \mathcal{F}$ whenever $i \in \mathbb{N}$ and $F = X$ for all but finitely many $i \in \mathbb{N}$.

Lemma

Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(\mathcal{F},\mathcal{G})$ is Cauchy filter pair on $(X,\mathcal{U})$, then $(\prod_{i\in\mathbb{N}}\mathcal{F},\prod_{i\in\mathbb{N}}\mathcal{G})$ is Cauchy filter pair on $\left(\prod_{i\in\mathbb{N}}X,\overline{\mathcal{U}}\right)$.

Lemma

Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(\mathcal{F},\mathcal{G})$ is Cauchy filter pair on $(X,\mathcal{U})$, then $(\prod_{i\in\mathbb{N}}\mathcal{F},\prod_{i\in\mathbb{N}}\mathcal{G})$ is Cauchy filter pair on $\left(\prod_{i\in\mathbb{N}}X,\overline{\mathcal{U}}\right)$.

Theorem

Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on the quasi-uniform box product $\left(\prod_{i\in\mathbb{N}}X,\overline{\mathcal{U}}\right)$ of $(X,\mathcal{U})$, then the filter pair $(\mathcal{F},\mathcal{G})$ defined by

$$\mathcal{F}=\left\{F\subseteq X:\prod_{i\in\mathbb{N}}F\in\overline{\mathcal{F}}\right\}$$

and

$$\mathcal{G}=\left\{G\subseteq X:\prod_{i\in\mathbb{N}}F\in\overline{\mathcal{G}}\right\},$$

is a Cauchy filter pair on $(X,\mathcal{U})$.

Completeness

It is well-known that a quasi-uniform space $(X,\mathcal{U})$ is *D*-complete provided that if $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair on $(X,\mathcal{U})$, then $\mathcal{G}$ converges with respect to $\tau(\mathcal{U})$.

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Theorem

If quasi-uniform space is D-complete, then its quasi-uniform box product is D-complete too.

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It is well-known that a quasi-uniform space $(X,\mathcal{U})$ is *D*-complete provided that if $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair on $(X,\mathcal{U})$, then $\mathcal{G}$ converges with respect to $\tau(\mathcal{U})$.

Theorem

If quasi-uniform space is D-complete, then its quasi-uniform box product is D-complete too.

Let $(X,\mathcal{U})$ be a quasi-uniform space. It is not difficult to prove that $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is bicomplete whenever $(X,\mathcal{U})$ is bicomplete. One can use ideas from Theorem above to prove that if $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$, then $\overline{\mathcal{F}}$ converges with respect to $\tau(\overline{\mathcal{U}^{-1}})$ and $\overline{\mathcal{G}}$ converges with respect to $\tau(\overline{\mathcal{U}})$.

Let $(X,\mathcal{U})$ be a quasi-uniform space. It is not difficult to prove that $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is bicomplete whenever $(X,\mathcal{U})$ is bicomplete. One can use ideas from Theorem above to prove that if $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$, then $\overline{\mathcal{F}}$ converges with respect to $\tau(\overline{\mathcal{U}^{-1}})$ and $\overline{\mathcal{G}}$ converges with respect to $\tau(\overline{\mathcal{U}})$. Furthermore, one can use the argument that if $(X,\mathcal{U})$ is bicomplete, then $(X,\mathcal{U}^s)$ is complete, therefore $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}^s\right)$ is complete as a uniform box product of $(X,\mathcal{U}^s)$. Hence $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is bicomplete.

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- Thank you
- Merci beaucoup
- Obrigado