Minimal homeomorphisms of a Cantor space: full groups and invariant measures

J. Melleray

Institut Camille Jordan (Université de Lyon)

Toposym 2016, Prague

Joint work with Tomás Ibarlucía (Lyon).

I. Full groups

Full groups in the topological context

K denotes a Cantor space; Γ, Δ are countable groups acting on K by homeomorphisms and *minimally*: all orbits are dense.

Full groups in the topological context

K denotes a Cantor space; Γ, Δ are countable groups acting on K by homeomorphisms and *minimally*: all orbits are dense.

Here we are particularly interested in the equivalence relation induced by the action of Γ on K. We denote by $[x]_\Gamma$ the Γ-orbit of $x \in K$.

Full groups in the topological context

K denotes a Cantor space; Γ, Δ are countable groups acting on K by homeomorphisms and *minimally*: all orbits are dense.

Here we are particularly interested in the equivalence relation induced by the action of Γ on K. We denote by $[x]_\Gamma$ the Γ-orbit of $x \in K$.

Definition

The full group $[\Gamma]$ is made up of all *homeomorphisms* g of K such that for all $x \in K$ there exists $\gamma \in \Gamma$ satisfying $\gamma x = gx$.

Full groups in the topological context

K denotes a Cantor space; Γ, Δ are countable groups acting on K by homeomorphisms and *minimally*: all orbits are dense.

Here we are particularly interested in the equivalence relation induced by the action of Γ on K. We denote by $[x]_\Gamma$ the Γ-orbit of $x \in K$.

Definition

The full group $[\Gamma]$ is made up of all *homeomorphisms* g of K such that for all $x \in K$ there exists $\gamma \in \Gamma$ satisfying $\gamma x = gx$.

That is, for all x one has $g([x]_\Gamma) = [x]_\Gamma$.

Full groups and orbit equivalence

Definition

The actions of Γ, Δ on K are *orbit equivalent* if there exists a homeomorphism h of K such that

$$\forall x \in K \quad h([x]_\Gamma) = [h(x)]_\Delta \ .$$

Full groups and orbit equivalence

Definition

The actions of Γ, Δ on K are *orbit equivalent* if there exists a homeomorphism h of K such that

$$\forall x \in K \quad h([x]_\Gamma) = [h(x)]_\Delta \ .$$

That is, the orbit partitions of K induced by the actions of Γ and Δ are isomorphic.

Full groups and orbit equivalence

Definition

The actions of Γ, Δ on K are *orbit equivalent* if there exists a homeomorphism h of K such that

$$\forall x \in K \quad h([x]_\Gamma) = [h(x)]_\Delta \ .$$

That is, the orbit partitions of K induced by the actions of Γ and Δ are isomorphic.

Theorem (Giordano–Putnam–Skau; Medynets)

Assume Γ, Δ act minimally on K and $\varphi \colon [\Gamma] \to [\Delta]$ is an isomorphism. Then there exists $g \in \mathrm{Homeo}(K)$ such that $\varphi(T) = gTg^{-1}$ for all $T \in [\Gamma]$.

Full groups and orbit equivalence

Definition

The actions of Γ, Δ on K are *orbit equivalent* if there exists a homeomorphism h of K such that

$$\forall x \in K \quad h([x]_\Gamma) = [h(x)]_\Delta \ .$$

That is, the orbit partitions of K induced by the actions of Γ and Δ are isomorphic.

Theorem (Giordano–Putnam–Skau; Medynets)

Assume Γ, Δ act minimally on K and $\varphi \colon [\Gamma] \to [\Delta]$ is an isomorphism. Then there exists $g \in \mathrm{Homeo}(K)$ such that $\varphi(T) = gTg^{-1}$ for all $T \in [\Gamma]$.
In particular, an isomorphism between full groups must come from an orbit equivalence (and conversely).

Full groups in the measurable setting

The situation we just described has a measurable counterpart, where one considers p.m.p actions on a standard probability space (X, μ), whose automorphism group we denote by $\mathrm{Aut}(X, \mu)$.

Full groups in the measurable setting

The situation we just described has a measurable counterpart, where one considers p.m.p actions on a standard probability space (X, μ), whose automorphism group we denote by $\mathrm{Aut}(X, \mu)$.

Definition

Given a countable p.m.p action of a countable group Γ on (X, μ), the full group $[\Gamma]_\mu$ is the subgroup of $\mathrm{Aut}(X, \mu)$ made up of all g such that for (almost) all x there exists γ satisfying $g(x) = \gamma x$.

Full groups in the measurable setting

The situation we just described has a measurable counterpart, where one considers p.m.p actions on a standard probability space (X, μ), whose automorphism group we denote by $\mathrm{Aut}(X, \mu)$.

Definition

Given a countable p.m.p action of a countable group Γ on (X, μ), the full group $[\Gamma]_\mu$ is the subgroup of $\mathrm{Aut}(X, \mu)$ made up of all g such that for (almost) all x there exists γ satisfying $g(x) = \gamma x$.

Theorem (Dye)

Given two countable groups Δ, Γ acting ergodically on (X, μ), and an isomorphism $\varphi \colon [\Gamma]_\mu \to [\Delta]_\mu$, there exists $g \in \mathrm{Aut}(X, \mu)$ such that $\varphi(T) = gTg^{-1}$ for all $T \in [\Gamma]_\mu$.

Topologies on $\mathrm{Aut}(X, \mu)$

$\mathrm{Aut}(X, \mu)$ is a Polish group when endowed with the topology τ induced by the maps $g \mapsto \mu(g(A) \Delta B)$.

Topologies on $\mathrm{Aut}(X,\mu)$

$\mathrm{Aut}(X,\mu)$ is a Polish group when endowed with the topology τ induced by the maps $g \mapsto \mu(g(A)\Delta B)$.

One could also endow $\mathrm{Aut}(X,\mu)$ with the uniform topology, coming from the metric

$$d_u(g,h) = \mu(\{x \colon g(x) \neq h(x)\}) \ .$$

The topology induced by d_u is very much non separable.

Topologies on $\mathrm{Aut}(X,\mu)$

$\mathrm{Aut}(X,\mu)$ is a Polish group when endowed with the topology τ induced by the maps $g \mapsto \mu(g(A)\Delta B)$.

One could also endow $\mathrm{Aut}(X,\mu)$ with the uniform topology, coming from the metric

$$d_u(g,h) = \mu(\{x\colon g(x) \neq h(x)\}) \ .$$

The topology induced by d_u is very much non separable.

$[\Gamma]_\mu$ is not a closed subset of $(\mathrm{Aut}(X,\mu),\tau)$; when the action is ergodic $[\Gamma]_\mu$ is dense in $\mathrm{Aut}(X,\mu)$.

Topologies on $\mathrm{Aut}(X, \mu)$

$\mathrm{Aut}(X, \mu)$ is a Polish group when endowed with the topology τ induced by the maps $g \mapsto \mu(g(A) \Delta B)$.

One could also endow $\mathrm{Aut}(X, \mu)$ with the uniform topology, coming from the metric

$$d_u(g, h) = \mu(\{x \colon g(x) \neq h(x)\}) \ .$$

The topology induced by d_u is very much non separable.

$[\Gamma]_\mu$ is not a closed subset of $(\mathrm{Aut}(X, \mu), \tau)$; when the action is ergodic $[\Gamma]_\mu$ is dense in $\mathrm{Aut}(X, \mu)$.

At least, $[\Gamma]_\mu$ is a Borel subset of $\mathrm{Aut}(X, \mu)$ (Wei).

Uniqueness of the Polish topology for measured full groups

$[\Gamma]_\mu$ is a closed subgroup of $(\mathrm{Aut}(X,\mu), d_u)$, and the induced topology turns $[\Gamma]_\mu$ into a Polish group.

Uniqueness of the Polish topology for measured full groups

$[\Gamma]_\mu$ is a closed subgroup of $(\mathrm{Aut}(X,\mu), d_u)$, and the induced topology turns $[\Gamma]_\mu$ into a Polish group.

Theorem (Kittrell–Tsankov)

Whenever the action of Γ on (X,μ) is ergodic, its full group has the automatic continuity property: any homomorphism from $[\Gamma]_\mu$ to a separable group is continuous.

Uniqueness of the Polish topology for measured full groups

$[\Gamma]_\mu$ is a closed subgroup of $(\mathrm{Aut}(X,\mu), d_u)$, and the induced topology turns $[\Gamma]_\mu$ into a Polish group.

Theorem (Kittrell–Tsankov)

Whenever the action of Γ on (X,μ) is ergodic, its full group has the automatic continuity property: any homomorphism from $[\Gamma]_\mu$ to a separable group is continuous.

So the Polish topology of $[\Gamma]_\mu$ is completely encoded in its algebraic structure when the action is ergodic.

Rise and Fall of a theorem

The group Homeo(K) also has a natural Polish topology (given by the sup-metric, or equivalently by viewing it as a subgroup of the group of permutations of all clopen sets).

Rise and Fall of a theorem

The group Homeo(K) also has a natural Polish topology (given by the sup-metric, or equivalently by viewing it as a subgroup of the group of permutations of all clopen sets).

Obviously true Theorem

Whenever Γ is a countable group acting minimally on a Cantor space, the full group $[\Gamma]$ satisfies the automatic continuity property for its natural Polish topology.

Rise and Fall of a theorem

The group Homeo(K) also has a natural Polish topology (given by the sup-metric, or equivalently by viewing it as a subgroup of the group of permutations of all clopen sets).

Obviously true Theorem

Whenever Γ is a countable group acting minimally on a Cantor space, the full group $[\Gamma]$ satisfies the automatic continuity property for its natural Polish topology.

Minor concern

... What is this natural Polish topology, by the way?

The search was futile

Theorem (Ibarlucía–M.)

There is no second-countable, Baire, Hausdorff group topology on $[\Gamma]$.

The search was futile

Theorem (Ibarlucía–M.)

There is no second-countable, Baire, Hausdorff group topology on $[\Gamma]$.

Theorem (Ibarlucía–M.)

Even worse: any Baire, Hausdorff group topology on $[\Gamma]$ must refine the topology induced from the Polish topology on $\mathrm{Homeo}(K)$; yet...

The search was futile

Theorem (Ibarlucía–M.)

There is no second-countable, Baire, Hausdorff group topology on $[\Gamma]$.

Theorem (Ibarlucía–M.)

Even worse: any Baire, Hausdorff group topology on $[\Gamma]$ must refine the topology induced from the Polish topology on $\mathrm{Homeo}(K)$; yet...

Whenever φ is a minimal homeomorphism of a Cantor space K, the full group $[\varphi]$ is a coanalytic non Borel subset of $\mathrm{Homeo}(K)$.

The search was futile

Theorem (Ibarlucía–M.)

There is no second-countable, Baire, Hausdorff group topology on $[\Gamma]$.

Theorem (Ibarlucía–M.)

Even worse: any Baire, Hausdorff group topology on $[\Gamma]$ must refine the topology induced from the Polish topology on $\mathrm{Homeo}(K)$; yet...

Whenever φ is a minimal homeomorphism of a Cantor space K, the full group $[\varphi]$ is a coanalytic non Borel subset of $\mathrm{Homeo}(K)$.

The proof uses a result of Glasner and Weiss: whenever A, B are clopen subsets such that $\mu(A) = \mu(B)$ for any φ-invariant measure μ, there exists $g \in [\varphi]$ such that $g(A) = B$.

II. Closures of full groups

Characterization of the closure of $[\varphi]$

Theorem (Glasner–Weiss)

Assume φ is a minimal homeomorphism of K; denote by X_φ the set of all probability measures on K preserved by φ. Then the closure of $[\varphi]$ in Homeo(K) is

$$G_\varphi = \{g \colon \forall \mu \in X_\varphi \ g^*\mu = \mu\} \ .$$

Characterization of the closure of $[\varphi]$

Theorem (Glasner–Weiss)

Assume φ is a minimal homeomorphism of K; denote by X_φ the set of all probability measures on K preserved by φ. Then the closure of $[\varphi]$ in Homeo(K) is

$$G_\varphi = \{g \colon \forall \mu \in X_\varphi \ g^*\mu = \mu\} \ .$$

Theorem (essentially Giordano–Putnam–Skau)

If G_φ and G_ψ are isomorphic then φ and ψ are orbit equivalent.
(This follows from a GPS theorem stating that φ, ψ are orbit equivalent as soon as $X_\varphi = X_\psi$)

Characterization of the closure of $[\varphi]$

Theorem (Glasner–Weiss)

Assume φ is a minimal homeomorphism of K; denote by X_φ the set of all probability measures on K preserved by φ. Then the closure of $[\varphi]$ in Homeo(K) is

$$G_\varphi = \{g \colon \forall \mu \in X_\varphi \ g^*\mu = \mu\} .$$

Theorem (essentially Giordano–Putnam–Skau)

If G_φ and G_ψ are isomorphic then φ and ψ are orbit equivalent.
(This follows from a GPS theorem stating that φ, ψ are orbit equivalent as soon as $X_\varphi = X_\psi$)

We do not know whether G_φ has the automatic continuity property (at least its Polish group topology is unique).

How little we know.

- Is G_φ simple? What about $[\varphi]$? (both are *topologically* simple for the topology induced by $\mathrm{Homeo}(K)$)

How little we know.

- Is G_φ simple? What about $[\varphi]$? (both are *topologically* simple for the topology induced by Homeo(K))
- Does Glasner–Weiss' characterization of the closure of the full group of a minimal homeomorphism remain true for minimal actions of *amenable* groups?

How little we know.

- Is G_φ simple? What about $[\varphi]$? (both are *topologically* simple for the topology induced by $\mathrm{Homeo}(K)$)
- Does Glasner–Weiss' characterization of the closure of the full group of a minimal homeomorphism remain true for minimal actions of *amenable* groups?
- If Γ, Δ are amenable and $\overline{[\Gamma]} \cong \overline{[\Delta]}$, are the actions of Γ and Δ orbit-equivalent?

How little we know.

- Is G_φ simple? What about $[\varphi]$? (both are *topologically* simple for the topology induced by Homeo(K))
- Does Glasner–Weiss' characterization of the closure of the full group of a minimal homeomorphism remain true for minimal actions of *amenable* groups?
- If Γ, Δ are amenable and $\overline{[\Gamma]} \cong \overline{[\Delta]}$, are the actions of Γ and Δ orbit-equivalent?

  The last question appears completely out of reach in this generality. Related to the last two:

- Given a simplex X of probability measures on K, when does there exist a minimal homeomorphism φ of K such that $X = X_\varphi$?

How little we know.

- Is G_φ simple? What about $[\varphi]$? (both are *topologically* simple for the topology induced by Homeo(K))
- Does Glasner–Weiss' characterization of the closure of the full group of a minimal homeomorphism remain true for minimal actions of *amenable* groups?
- If Γ, Δ are amenable and $\overline{[\Gamma]} \cong \overline{[\Delta]}$, are the actions of Γ and Δ orbit-equivalent?

  The last question appears completely out of reach in this generality. Related to the last two:

- Given a simplex X of probability measures on K, when does there exist a minimal homeomorphism φ of K such that $X = X_\varphi$?
  A result of Akin answers that question for X a singleton, and unpublished work of Dahl extends that to the finite-dimensional case.

III.Invariant measures.

Necessary conditions

For X to coincide with the set of invariant measures of some minimal homeomorphism,

Necessary conditions

For X to coincide with the set of invariant measures of some minimal homeomorphism,

- X must be nonempty, compact, and convex (even, a *Choquet simplex*).

Necessary conditions

For X to coincide with the set of invariant measures of some minimal homeomorphism,

- X must be nonempty, compact, and convex (even, a *Choquet simplex*).
- All elements of X must be atomless and with full support.

Necessary conditions

For X to coincide with the set of invariant measures of some minimal homeomorphism,

- X must be nonempty, compact, and convex (even, a *Choquet simplex*).
- All elements of X must be atomless and with full support.
- X must be *good*: whenever A, B are clopen and $\forall \mu \in X \ \mu(A) < \mu(B)$,

Necessary conditions

For X to coincide with the set of invariant measures of some minimal homeomorphism,

- X must be nonempty, compact, and convex (even, a *Choquet simplex*).
- All elements of X must be atomless and with full support.
- X must be *good*: whenever A, B are clopen and $\forall \mu \in X \ \mu(A) < \mu(B)$, $\exists C \subset B$ clopen s.t. $\forall \mu \in X \ \mu(C) = \mu(A)$.

Necessary conditions

For X to coincide with the set of invariant measures of some minimal homeomorphism,

- X must be nonempty, compact, and convex (even, a *Choquet simplex*).
- All elements of X must be atomless and with full support.
- X must be *good*: whenever A, B are clopen and $\forall \mu \in X \ \mu(A) < \mu(B)$, $\exists C \subset B$ clopen s.t. $\forall \mu \in X \ \mu(C) = \mu(A)$.

To explain another necessary condition, let us recall the concept of a *Kakutani–Rokhlin partition*.

Kakutani–Rokhlin partitions: in words

Let φ be a minimal homeomorphism of K. Then for any nonempty clopen B there exists $n \geq 1$ such that

$$K = \bigcup_{k=1}^{n} \varphi^k(B) \; .$$

Kakutani–Rokhlin partitions: in words

Let φ be a minimal homeomorphism of K. Then for any nonempty clopen B there exists $n \geq 1$ such that

$$K = \bigcup_{k=1}^{n} \varphi^k(B) .$$

Given $x \in B$, let $k_x = \min\{k \geq 1 \colon \varphi^k(x) \in B\}$ and

Kakutani–Rokhlin partitions: in words

Let φ be a minimal homeomorphism of K. Then for any nonempty clopen B there exists $n \geq 1$ such that

$$K = \bigcup_{k=1}^{n} \varphi^k(B) \ .$$

Given $x \in B$, let $k_x = \min\{k \geq 1 \colon \varphi^k(x) \in B\}$ and

$$B_k = \{x \in B \colon k_x = k\} \quad B_{k,i} = \varphi^i(B_k) \quad (0 \leq i \leq k-1) \ .$$

Kakutani–Rokhlin partitions: in words

Let φ be a minimal homeomorphism of K. Then for any nonempty clopen B there exists $n \geq 1$ such that

$$K = \bigcup_{k=1}^{n} \varphi^k(B) \ .$$

Given $x \in B$, let $k_x = \min\{k \geq 1 \colon \varphi^k(x) \in B\}$ and

$$B_k = \{x \in B \colon k_x = k\} \quad B_{k,i} = \varphi^i(B_k) \quad (0 \leq i \leq k-1) \ .$$

Then $K = \bigsqcup B_{k,i}$ is the Kakutani–Rokhlin partition associated to B, φ.

Kakutani–Rokhlin partitions: in pictures

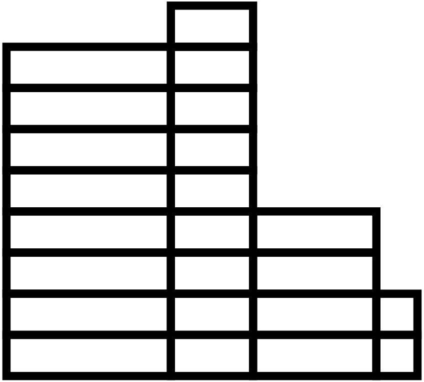

Figure: A KR partition

Kakutani–Rokhlin partitions: in pictures

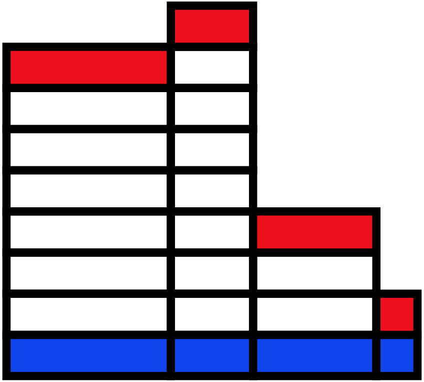

Figure: The base appears in blue and the *top* in red

Kakutani–Rokhlin partitions: in pictures

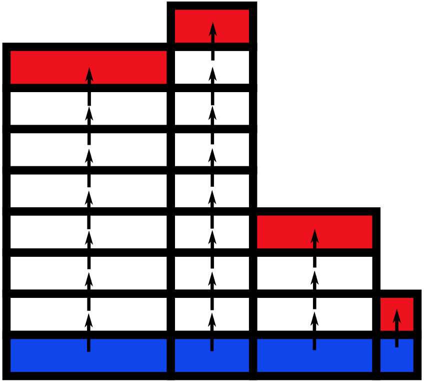

Figure: The action on atoms of the tower off the top is prescribed

Approximate divisibility

Definition (M.–Ibarlucía)

Let X be a set of probability measures on K. Then X is *approximately divisible* if for all n, all $\varepsilon > 0$ and any clopen A there exists a clopen $B \subseteq A$ such that

$$\forall \mu \in K \ \mu(A) - \varepsilon \leq n\mu(B) \leq \mu(A) \ .$$

Approximate divisibility

Definition (M.–Ibarlucía)

Let X be a set of probability measures on K. Then X is *approximately divisible* if for all n, all $\varepsilon > 0$ and any clopen A there exists a clopen $B \subseteq A$ such that

$$\forall \mu \in K \ \mu(A) - \varepsilon \leq n\mu(B) \leq \mu(A) \ .$$

Proposition (M.–Ibarlucía)

If $X = X_\varphi$ for some minimal φ then X is approximately divisible.

Simplices of invariant measures are approximately divisible

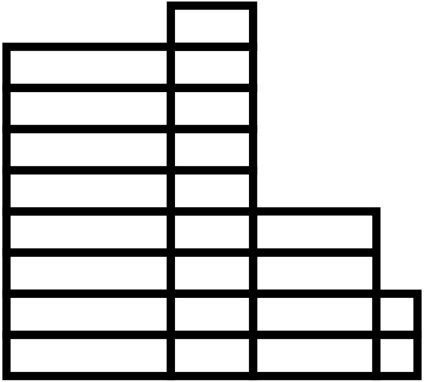

Figure: A KR partition with a small base B.

Simplices of invariant measures are approximately divisible

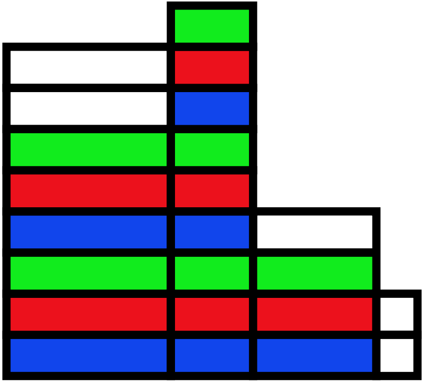

Figure: 3 pieces of equal measures, plus a rest with measures $< 2\mu(B)$.

A characterization

Theorem (Ibarlucía–M.)

Let X be a subset of the space of probability measures on a Cantor space K. There exists a minimal $\varphi \in \text{Homeo}(K)$ such that $X = \{\mu \colon \varphi^* \mu = \mu\}$ iff

- X is nonempty, compact, and convex.

A characterization

Theorem (Ibarlucía–M.)

Let X be a subset of the space of probability measures on a Cantor space K. There exists a minimal $\varphi \in \text{Homeo}(K)$ such that $X = \{\mu \colon \varphi^*\mu = \mu\}$ iff

- X is nonempty, compact, and convex.
- All elements of X are atomless and with full support.

A characterization

Theorem (Ibarlucía–M.)

Let X be a subset of the space of probability measures on a Cantor space K. There exists a minimal $\varphi \in \mathrm{Homeo}(K)$ such that $X = \{\mu \colon \varphi^* \mu = \mu\}$ iff

- X is nonempty, compact, and convex.
- All elements of X are atomless and with full support.
- X is good.

A characterization

Theorem (Ibarlucía–M.)

Let X be a subset of the space of probability measures on a Cantor space K. There exists a minimal $\varphi \in \text{Homeo}(K)$ such that $X = \{\mu \colon \varphi^*\mu = \mu\}$ iff

- X is nonempty, compact, and convex.
- All elements of X are atomless and with full support.
- X is good.
- X is approximately divisible.

A characterization

Theorem (Ibarlucía–M.)

Let X be a subset of the space of probability measures on a Cantor space K. There exists a minimal $\varphi \in \mathrm{Homeo}(K)$ such that $X = \{\mu \colon \varphi^* \mu = \mu\}$ iff

- X is nonempty, compact, and convex.
- All elements of X are atomless and with full support.
- X is good.
- X is approximately divisible.

When X is finite-dimensional the last assumption is redundant; unknown in general. The result for X a singleton is due to Akin, and the f.d. case (with a mild additional assumption) to Dahl.

Observation (M.–Tsankov)

Whenever Γ is a f.g countable group acting freely and minimally on a Cantor space, the simplex of all Γ-invariant measures is approximately divisible.

Observation (M.–Tsankov)

Whenever Γ is a f.g countable group acting freely and minimally on a Cantor space, the simplex of all Γ-invariant measures is approximately divisible.

Theorem (M.–Tsankov)

Let Γ be a f.g nilpotent group acting freely minimally on a Cantor space K; then there exists a minimal homeomorphism φ of K such that

$$\{\mu\colon \forall\gamma \in \Gamma\ \gamma^*\mu = \mu\} = \{\mu\colon \varphi^*\mu = \mu\}\ .$$

Observation (M.–Tsankov)

Whenever Γ is a f.g countable group acting freely and minimally on a Cantor space, the simplex of all Γ-invariant measures is approximately divisible.

Theorem (M.–Tsankov)

Let Γ be a f.g nilpotent group acting freely minimally on a Cantor space K; then there exists a minimal homeomorphism φ of K such that

$$\{\mu\colon \forall \gamma \in \Gamma \ \gamma^*\mu = \mu\} = \{\mu\colon \varphi^*\mu = \mu\} \ .$$

To obtain this result for nilpotent groups, we apply deep, hard work of Schneider–Seward, itself building upon deep, hard work of Gao–Jackson in the abelian case. It is a weak positive answer to the question of whether any minimal action of a nilpotent group is orbit equivalent to a minimal $\mathbb{Z}$-action.

Thank you for your attention!