Hereditary Interval Algebras

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Interval Algebras

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Example

Given a linearly ordered set $(L, <)$ define $Int(L)$ by

$$\{[a_0, b_0) \cup \ldots \cup [a_n, b_n) : -\infty \leq a_0 < b_0 < \ldots < a_n < b_n \leq \infty \in L\}.$$

Interval Algebras

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Theorem (Rubin, 1983)

If A is a subalgebra of $Int(L)$, of cardinality κ uncountable regular, then A has a chain or an anti chain of size κ.

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The class of interval algebras is closed under homomorphic images and finite products.

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An interval algebra is *hereditary* if every subalgebra is an interval algebra.

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Theorem (Bekkali-Todorcevic, 2015)

Every hereditary interval algebra is σ-centered.

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Theorem (Nikiel, Purisch, Treybig, 1998)

There is σ-centered interval algebra of cardinality continuum that is not hereditary.

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Theorem (Bekkali-Todorcevic, 2015)

Every subalgebra of a σ-centered subalgebra of cardinality $< \mathfrak{b}$ is an interval algebra itself. In particular, every interval algebra over a set of reals of cardinality $< \mathfrak{b}$ is hereditary.

Cardinal Invariants

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Remark

It follows from the previous theorems that $\mathfrak{b} \leq \mu \leq non(\mathcal{M})$.

Cardinal Invariants

For each $f \in 2^{\mathbb{Q}}$, let τ_f be the topology over $[0,1]$ where every irrational has its usual neighborhood basis and a basic neighborhood of a point $q \in \mathbb{Q}$ is of the form $[q, q + \frac{1}{n})$ if $f(q) = 0$ and $(-\frac{1}{n} + q, q]$ if $f(q) = 1$.

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Definition

Let $A(X) = I \times \{0\} \cup (\mathbb{Q} \cup X) \times \{1\} \cup \mathbb{Q} \times \{2\}$ with the topology given by the lexicographical order.

Cardinal Invariants

Theorem (M-R)

If $\mathbb{Q}$ is not relatively G_δ in $X \cup \mathbb{Q}$ in the τ_f topology, for all $f \in 2^{\mathbb{Q}}$. Then clop$(A(X))$ is not hereditary.

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Theorem (M-R)

It is consistent with ZFC that $\mu <$ non$(\mathcal{M})$.

Sketch of proof

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4. $f : Q \to Q \cup X$ without fixed points s.t. $f(q)$ is a jump and $q \in (f(q), f(q)^+)_\prec$ or $q \in (f(q)^+, f(q))_\prec$.

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5. $A \subset Q$ then $A' = f(A) \cup f(A)'$.
6. $\forall x \in X \ F_x = \{q : x \in (q, f(q))\}$ is finite.
7. $X_N = \{x \in X : |F_x| = N\}$ es G_δ in the τ_f topology.

Thank you!