Notes on free topological (Abelian) topological groups

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Definitions

G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *topological group*.

Let $\sigma : X \to G$ be a continuous mapping of space X to a Hausdorff topological group G that satisfies the following conditions:

1) The image $\sigma(X)$ topologically generates the group G;

2) for every continuous mapping $f : X \to H$ to a topological group H, there exists a continuous homomorphism $\tilde{f} : G \to H$ such that $\tilde{f} \circ \sigma = f$.

Then the triple (G, X, σ) is denoted by $F(X)$ and is called the free topological group on X.

If all the groups in the above definition are Abelian, the triple (G, X, σ) is said to be the free Abelian topological group on X, and we designate $A(X)$.

X generates the free group $F_a(X)$, each element $g \in F_a(X)$ has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \cdots, x_n \in X$ and $\varepsilon_1, \cdots, \varepsilon_n = \pm 1$. This word for g is called *reduced* if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$. If the word g is reduced and non-empty, then it is different from the neutral element of $F_a(X)$. In particular, each element $g \in F_a(X)$ distinct from the neutral element can be uniquely written in the form $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$, where $n \geq 1$, $\varepsilon_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_{i+1}$ for each $i = 1, \cdots, n-1$. For every non-negative integer n, denote by $F_n(X)$ and $A_n(X)$ the subspace of topological group $F(X)$ and $A(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X, respectively.

The following results are well known.

Theorem If the free (Abelian) topological group $F(X)(A(X))$ is first-countable, then X is discrete.

Theorem If the free (Abelian) topological group $F(X)(A(X))$ is Fréchet-Urysohn, then X is discrete.

Theorem Either every convergent sequence of $F(X)(A(X))$ is finite or $F(X)(A(X))$ contains a copy of S_ω, equivalently, S_2.

Theorem If $F(X)(A(X))$ is a q-space, then X is discrete.

Theorem If $F(X)(A(X))$ is κ-Fréchet-Urysohn, then X is discrete.

Theorem (Yamada) Let X be a metrizable space. If $F_5(X)$ is Fréchet-Urysohn, then X is compact or discrete.

Theorem Let X be a topological space in which the closure of a bounded subset in X is compact. If $F_5(X)$ is Fréchet-Urysohn, then X is compact or discrete.

Theorem (Arhangel'skii, Okunev and Pestov) Let X be a metrizable space. $A(X)$ is a k-space if and only if X is locally compact and NI(X) is separable.

Theorem (Yamada) Let X be a metric space, $F(X)$ is a k-space if and only if $F_n(X)$ is a k-space for each n.

Theorem (Yamada) Let X be a metrizable space. Then $A_n(X)$ is a k-space for each n if and only if $A_4(X)$ is a k-space.

Question: Let X be a metrzable space, if $A_n(X)$ is a k-space for each n, is $A(X)$ a k-space?

Theorem(Yamada) Let X be the first-countable hedgehog space with countable many spines. $A_n(X)$ is a k-space for each n, but $A(X)$ is not a k-space.

Question Let X be a metrizable space, if $F_i(X)$ is a k-space, where $i = 4, 5, 6, 7$, is $F(X)$ a k-space?

Theorem Let X be a non-metrizable, Lašnev space. Then the following are equivalent.

1. $A(X)$ is a k-space.

2. $A_n(X)$ is a k-space for each n.

3. $A_4(X)$ is a k-space.

4. X is a topological sum of a k-space with a countable k-network consisting of compact subsets and a discrete space.

Define the quasi-order $\leq^*$ on ${}^\omega\omega$ by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A subset of ${}^\omega\omega$ is called unbounded if it is unbounded in $< {}^\omega\omega, \leq^* >$. $\flat = min\{|B| : B$ is an unbounded subset of ${}^\omega\omega\}$.

Theorem Assume $\flat = \omega_1$. For a non-metrizable Lašnev spaces X, $A_3(X)$ is a sequential space if and only if $A(X)$ is a sequential space.

Theorem Assume $\flat > \omega_1$. There exists a non-metrizable Lašnev space X such that $A_3(X)$ is a sequential space but $A(X)$ is not.

Theorem Let X be a Lašnev space. $A_2(X)$ is a k-space if and only if X is metrizable or X is a topological sum of k_ω-subspaces.

$M_3 = \bigoplus\{C_\alpha : \alpha < \omega_1\}$, where $C_\alpha = \{x(n,\alpha) : n \in \mathbb{N}\} \cup \{x_\alpha\}$, $x(n,\alpha) \to x_\alpha$.

Let $X = S_\omega \bigoplus M_3$, it is easy to see that, in ZFC, $A_2(X)$ is a sequential space, but $A(X)$ is not.

Future Work

Thank You