ω^ω-bases in free objects over uniform spaces

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Papers to be published

All reported results are based on the following works by

[LPT] A. Leiderman, V. Pestov and A. Tomita
On topological groups admitting a base at identity indexed with ω^ω, 2015 (submitted for publication);

[BL1] T. Banakh and A. Leiderman
Local ω^ω-bases in free (locally convex) topological vector spaces, 2016 (submitted for publication);

[BL2] T. Banakh and A. Leiderman
Local ω^ω-bases in free (Abelian) topological groups, 2016, preprint;
[Ban] T. Banakh
ω^ω-bases in topological and uniform spaces, 2016, preprint.

1. Background and Motivation

In this talk we consider only Tychonoff topological spaces.
For a Tychonoff topological space X by $\mathbb{U}(X)$ we denote *the universal uniformity* of X, i.e., the universal uniformity, compatible with the topology of X. It is generated by the base consisting of entourages $[\rho]_{<1} := \{(x, y) \in X \times X : \rho(x, y) < 1\}$ where ρ runs over all continuous pseudometrics on X.

Definition of $\mathsf{A}(X), \mathsf{B}(X)$

For a uniform space X its *free Abelian topological group* is a pair $(\mathsf{A}_u(X), \delta_X)$ consisting of an Abelian topological group $\mathsf{A}_u(X)$ and a uniformly continuous map $\delta_X : X \to \mathsf{A}_u(X)$ such that for every uniformly continuous map $f : X \to H$ into an Abelian topological group H there exists a continuous group homomorphism $\bar{f} : \mathsf{A}_u(X) \to H$ such that $\bar{f} \circ \delta_X = f$.
Replacing the adjective "Abelian" by "Boolean" in this definition, we get the definition of a *free Boolean topological group* $(\mathsf{B}_u(X), \delta_X)$ over a uniform space X.
For a Tychonoff space X the free Abelian and Boolean topological groups $\mathsf{A}(X)$ and $\mathsf{B}(X)$ coincide with the free Abelian and Boolean topological groups $\mathsf{A}_u(X)$ and $\mathsf{B}_u(X)$ of the space X endowed with the universal uniformity $\mathbb{U}(X)$.

Definition of $\mathrm{F}(X)$

For a uniform space X its *free topological group* is a pair $(\mathrm{F}_u(X), \delta_X)$ consisting of a topological group $\mathrm{F}_u(X)$ and a uniformly continuous map $\delta_X : X \to \mathrm{F}_u(X)$ such that for every uniformly continuous map $f : X \to H$ into a topological group H there exists a continuous group homomorphism $\bar{f} : \mathrm{A}_u(X) \to H$ such that $\bar{f} \circ \delta_X = f$.
For a Tychonoff space X the free topological groups $\mathrm{F}(X)$ coincides with the free topological group $\mathrm{F}_u(X)$ of the space X endowed with the universal uniformity $\mathbb{U}(X)$.

Definition of $\mathsf{L}(X), \mathsf{V}(X)$

For a uniform space X its *free locally convex space* is a pair $(\mathsf{L}_u(X), \delta_X)$ consisting of a locally convex space $\mathsf{L}_u(X)$ and a uniformly continuous map $\delta_X : X \to \mathsf{L}_u(X)$ such that for every uniformly continuous map $f : X \to Y$ into a locally convex space Y there exists a continuous linear operator $\bar{f} : \mathsf{L}_u(X) \to Y$ such that $\bar{f} \circ \delta_X = f$.
For a Tychonoff topological space X the free locally convex space $\mathsf{L}(X)$ coincides with the free locally convex space $\mathsf{L}_u(X)$ of the space X endowed with the universal uniformity $\mathbb{U}(X)$.

Deleting "locally convex", we obtain the definition of a *free topological vector space* $\mathsf{V}(X)$ of X.

It is well-known that for every Tychonoff topological space X its free objects $\mathrm{A}(X), \mathrm{B}(X), \mathrm{F}(X), \mathrm{L}(X), \mathrm{V}(X)$ exist and are unique up to a topological isomorphism.
It is worth mentioning that the canonical map δ_X is a closed topological embedding, so we can identify the space X with its image $\delta_X(X)$ and say that X algebraically generates all free topological objects.

Some known facts

- A(X) is a quotient topological group of F(X);
- B(X) is a quotient topological group of A(X);
- L(X) and V(X) algebraically coincide, and the topology of V(X) is finer than the topology of L(X);
- A(X) naturally embeds into L(X);
- If X is a k-space, then L(X) naturally embeds into the double function space $C_k(C_k(X))$

Metrizability of free objects

- For an infinite Tychonoff space X the free spaces $\mathsf{L}(X)$ and $\mathsf{V}(X)$ are not first-countable (so do not admit neighborhood bases indexed by ω), therefore not metrizable.
- Free topological groups over X are metrizable iff X is discrete.
- (P. Nickolas and M. Tkachenko) If X is an infinite compact metrizable space, then the character $\chi(\mathsf{A}(X)) = \chi(\mathsf{F}(X)) = \mathfrak{d}$.

Definition of ω^ω-indexed local base

A topological space X is defined to have a neighborhood *ω^ω-base* at a point $x \in X$ if there exists a neighborhood base $(U_\alpha)_{\alpha \in \omega^\omega}$ at x such that $U_\beta \subset U_\alpha$ for all elements $\alpha \leq \beta$ in ω^ω.

We shall say that a topological space has a *local ω^ω-base* if it has an ω^ω-base at each point $x \in X$. Evidently, a topological group G has a *local ω^ω-base* if it has a neighborhood ω^ω-base at the identity $e \in G$.

A uniformity $\mathbb{U}(X)$ on a space X is defined to have an *ω^ω-base* if there is a base of entourages $\{U_\alpha\}_{\alpha \in \omega^\omega} \subset \mathbb{U}(X)$ such that $U_\beta \subset U_\alpha$ for all $\alpha \leq \beta$ in ω^ω.

B. Cascales and J. Orihuela were the first who considered uniform spaces admitting an ω^ω-base. They proved that compact spaces with this property are metrizable. For the first time the concept of an ω^ω-base appeared as a tool for studying locally convex spaces that belong to the class $\mathfrak{G}$ introduced by Cascales and Orihuela. Previously local ω^ω-bases were named $\mathfrak{G}$-bases. We change the name to a more natural one.

Published results

A topological space X is called a *cosmic k_ω-space* if there are metrizable compact subspaces $\{X_n\}_{n\in\omega}$ covering X such that $V \subseteq X$ is open in X if and only if $V \cap X_n$ is open in X_n for every $n \in \omega$.
For a cosmic k_ω-*space* X (in particular, for any metrizable compact X), it was shown earlier that

(a) (2015) A(X) and L(X) have a local ω^ω-base;
(b) (2015) F(X) has a local ω^ω-base.

General Problem

Our research concentrates on the following problem:
characterize those uniform/ Tychonoff spaces X such that

free topological groups $F(X)$, free Abelian topological groups $A(X)$, free Boolean topological groups $B(X)$; and

free locally convex space $L(X)$, free topological vector space $V(X)$

admit a local ω^ω-base.

2. Methods of Solution

- Find and use explicite description of the topology of free objects. In the case of A(X) and B(X) such description is known and is easy to use. In the case of F(X) known description appeared to be less applicable and we developed some new modificatons. In the case of L(X) and V(X) we found apparently new and completely satisfactory internal description of the topology.
- We used reductions and the relation of dominating between various partially ordered sets.

Reducibility of posets

Given two posets P, Q, we shall say that a subset $D \subset Q$ is *P-dominated in Q* if there exists a monotone map $f : P \to Q$ such that for every $x \in D$ there exists $y \in P$ with $x \leq f(y)$. It follows that a poset Q is P-dominated in Q if and only if Q reduces to P i.e. there exists a monotone cofinal map $f : P \to Q$. This kind of reducibility of posets is a bit stronger than the Tukey reducibility $\leq_T$, which requires the existence of a function $f : P \to Q$ which maps cofinal subsets of P to cofinal subsets of Q.

3. Solution for $A(X)$ and $B(X)$, where X is uniform

Theorem 1

For a uniform space X the following conditions are equivalent:

1. The free Abelian topological group $A_u(X)$ has a local ω^ω-base.
2. The free Boolean topological group $B_u(X)$ has a local ω^ω-base.
3. The uniformity $\mathbb{U}(X)$ has an ω^ω-base.

3. Solution for A(X) and B(X), where X is Tychonoff

Theorem 2

For a Tychonoff space X the following conditions are equivalent:

1. The free Abelian topological group A(X) of X has a local ω^ω-base.
2. The free Boolean topological group B(X) of X has a local ω^ω-base.
3. The universal uniformity $\mathbb{U}(X)$ of X has an ω^ω-base.

3. Solution for A(X) and B(X), where X is metrizable

Let us mention the relevant characterization by Ginsburg of Tychonoff spaces X whose universal uniformity $\mathbb{U}(X)$ has a countable base: those are exactly metrizable spaces with compact set of non-isolated points.

Theorem 3

For a metrizable space X the following conditions are equivalent:

1. The free Abelian topological group A(X) has a local ω^ω-base.
2. The free Boolean topological group B(X) has a local ω^ω-base.
3. the set X' of non-isolated points of X is σ-compact.

Example

Even for a countable space X with one non-isolated point, the topological group $A(X)$ need not have a ω^ω-base. Let ξ be any non-principal ultrafilter. We form the countable topological space $X = \omega \cup \{\xi\}$ considered as a subspace of $\beta\omega$ with the induced topology.

In fact, no non-principal ultrafilter is Tukey-reducible to ω^ω. Let D be a filter of subsets of ω and assume that $D \leq_T \omega^\omega$. Additionally, we view D as a subset of the Cantor set $\{0,1\}^\omega$. Thus D is a metric separable space with a partial order in which the set of predecessors of each element is compact. As a topological space the set ω^ω is of course analytic, and we deduce, by the work of S. Solecki and S. Todorcevic, that D is also analytic. However, it is well known that no non-principal ultrafilter is analytic.

Therefore the topological group $A(X)$ does not have a ω^ω-base.

4. Solution for $L_u(X)$ and $V(X)$, where X is uniform

Theorem 4

For a uniform space X the following conditions are equivalent:

1. the free locally convex space $L_u(X)$ has a local ω^ω-base;
2. the uniformity $\mathbb{U}(X)$ has an ω^ω-base and the poset $C_u(X)$ is ω^ω-dominated;
3. the uniformity $\mathbb{U}(X)$ has an ω^ω-base and the poset $C_u(X)$ is ω^ω-dominated in $\mathbb{R}^X$.

Theorem 5

Let X be a uniform space whose uniformity has an ω^ω-base. If the poset $C(X)$ is ω^ω-dominated in $\mathbb{R}^X$, then the free topological vector space $V_u(X)$ has a local ω^ω-base.

4. Solution for L(X) and V(X), where X is Tychonoff

Theorem 6

For a Tychonoff space X the following conditions are equivalent:

1. the free locally convex space L(X) of X has a local ω^ω-base;
2. the free topological vector space V(X) of X has a local ω^ω-base;
3. the universal uniformity $\mathbb{U}(X)$ of X has an ω^ω-base and the poset $C(X)$ is ω^ω-dominated;
4. the universal uniformity $\mathbb{U}(X)$ has an ω^ω-base and $C(X)$ is ω^ω-dominated in $\mathbb{R}^X$;
5. the universal uniformity $\mathbb{U}(X)$ has an ω^ω-base and the space X is a cosmic σ-compact space.

Theorem 7

For a Tychonoff space X

- if $\omega_1 < \mathfrak{b}$, then the conditions of Theorem 6 are equivalent to the property $(\omega\mathbb{U})$: the universal uniformity $\mathbb{U}(X)$ of X is ω-narrow and has an ω^ω-base;
- if $\omega_1 = \mathfrak{b}$, then the conditions of Theorem 6 are not equivalent to $(\omega\mathbb{U})$.

Theorem 8

For a metrizable space X the following conditions are equivalent:

1. the free locally convex space $\mathsf{L}(X)$ of X has a local ω^ω-base;
2. the free topological vector space $\mathsf{V}(X)$ of X has a local ω^ω-base;
3. X is σ-compact.

Inductive limits

A topological space X carries the *inductive topology with respect to a family* $(X_n)_{n\in\omega}$ of subsets of X if $V \subseteq X$ is open in X if and only if $V \cap X_n$ is open in X_n for every $n \in \omega$.

Proposition 9

Assume that a Tychonoff space X has the inductive topology with respect to a countable cover $\{X_n\}_{n\in\omega}$ of X.

- If for every $n \in \omega$ the group $\mathrm{A}(X_n)$ (resp. $\mathrm{B}(X_n)$) has a local ω^ω-base, then the group $\mathrm{A}(X)$ (resp. $\mathrm{B}(X)$) has a local ω^ω-base, too.
- If for every $n \in \omega$ the vector space $\mathrm{L}(X_n)$ (resp. $\mathrm{V}(X_n)$) has a local ω^ω-base, then the vector space $\mathrm{L}(X)$ (resp. $\mathrm{V}(X)$) has a local ω^ω-base, too.

Inductive limits

Proposition 10

Assume that a Tychonoff space X carries the inductive topology with respect to an increasing cover $(X_n)_{n\in\omega}$ of its metrizable σ-compact subspaces. Then the groups $\mathrm{A}(X)$ and $\mathrm{B}(X)$; and the vector spaces $\mathrm{L}(X)$ and $\mathrm{V}(X)$ all have local ω^ω-bases.

It is known that for any k-space holds: $\mathrm{L}(X) \subset C_k(C_k(X))$.

Theorem 11

If the space X is a k-space, then equivalent

- the free locally convex space $\mathrm{L}(X)$ of X has a local ω^ω-base;
- the double function space $C_k(C_k(X))$ has a local ω^ω-base.

5. Solution for $\mathsf{F}(X)$, where X is separable

Theorem 12

Let X be a separable uniform space. The following conditions are equivalent:

1. the free topological group $\mathsf{F}_u(X)$ on X admits a local ω^ω-base;
2. the uniformity $\mathbb{U}(X)$ has an ω^ω-base.

Theorem 13

Let X be a separable Tychonoff topological space. The following conditions are equivalent:

1. the free topological group $\mathsf{F}(X)$ on X admits a local ω^ω-base;
2. the universal uniformity $\mathbb{U}(X)$ has an ω^ω-base and the space X is a cosmic σ-compact space.

6. Solution for $\mathrm{F}(X)$, where X is not a P-space

Theorem 14

Let X be a Tychonoff space which is is not a P-space. The following conditions are equivalent:

1. the free topological group $\mathrm{F}(X)$ has a local ω^ω-base;
2. the subspace $Y = \{xyz^{-1}x^{-1} : x, y, z \in X\}$ of $\mathrm{F}(X)$ has a local ω^ω-base at the unit e of $\mathrm{F}(X)$;
3. the universal uniformity $\mathbb{U}(X)$ of X has an ω^ω-base and the function space $C(X)$ is ω^ω-dominated;
4. the universal uniformity $\mathbb{U}(X)$ has an ω^ω-base and the space X is a cosmic σ-compact space.

Theorem 15

Let X be a metrizable space. $\mathrm{F}(X)$ has a local ω^ω-base if and only if X is either discrete or σ-compact;

Example

Under $\omega_1 = \mathfrak{b}$ there exists an ω-narrow Tychonoff space X whose universal uniformity has an ω^ω-base but the free topological group $\mathrm{F}(X)$ fails to have a local ω^ω-base.

Theorem 16

Assume that $\mathfrak{b} = \mathfrak{d}$. Let X be a Tychonoff space, carrying the inductive topology with respect to an increasing cover $\{X_n\}_{n\in\omega}$. If for every $n \in \omega$ the free topological group $\mathrm{F}(X_n)$ has a local ω^ω-base, then the free topological group $\mathrm{F}(X)$ has a local ω^ω-base, too.

Problem

Is Theorem 16 true in ZFC?

Thank you!