Lelek fan and Poulsen simplex as Fraïssé limits

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- so the arrows are retractions onto K

Definitions - metric

- Assume that each $K \in \mathrm{Ob}(\mathcal{C})$ is equipped with a metric d_K.
- Given two $\mathcal{C}$-arrows $f, g \colon K \to L$, $f = \langle e, p \rangle$, $g = \langle i, q \rangle$, we define

$$d(f, g) = \begin{cases} \max_{y \in L} d_K(p(y), q(y)) & \text{if } e = i, \\ +\infty & \text{otherwise.} \end{cases}$$

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- $\mathcal{C}$ equipped with the metric d on each $\mathrm{Hom}(K, L)$ is a metric category if $d(f_0 \circ g, f_1 \circ g) \leq d(f_0, f_1)$ and $d(h \circ f_0, h \circ f_1) \leq d(f_0, f_1)$, whenever the composition makes sense.

Definitions - amalgamation

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- $\mathcal{C}$ has the almost amalgamation property if for every $\mathcal{C}$-arrows $f \colon A \to B$, $g \colon A \to C$, for every $\varepsilon > 0$, there exist $\mathcal{C}$-arrows $f' \colon B \to D$, $g' \colon C \to D$ such that $d(f' \circ f, g' \circ g) < \varepsilon$.

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- $\mathcal{C}$ has the strict amalgamation property if we can have f' and g' as above satisfying $f' \circ f = g' \circ g$.

Definitions - separability

$\mathcal{C}$ is separable if there is a countable subcategory $\mathcal{F}$ such that

(1) for every $X \in \text{Ob}(\mathcal{C})$ there are $A \in \text{Ob}(\mathcal{F})$ and a $\mathcal{C}$-arrow $f\colon X \to A$;

(2) for every $\mathcal{C}$-arrow $f\colon A \to Y$ with $A \in \mathrm{Ob}(\mathcal{F})$, for every $\varepsilon > 0$ there exists an $\mathcal{C}$-arrow $g\colon Y \to B$ and an $\mathcal{F}$-arrow $u\colon A \to B$ such that $d(g \circ f, u) < \varepsilon$.

Definitions - Fraïssé sequence

$\mathcal{C}$-sequence $\vec{U} = \langle U_m; u_m^n \rangle$ is a Fraïssé sequence if the following holds:

(F) Given $\varepsilon > 0$, $m \in \omega$, and an arrow $f\colon U_m \to F$, where $F \in \mathrm{Ob}(\mathcal{C})$, there exist $m < n$ and an arrow $g\colon F \to U_n$ such that $d(g \circ f, u_m^n) < \varepsilon$.

Criterion for a Fraïssé sequence

Theorem (Kubiś)

Let $\mathcal{C}$ be a directed metric category with objects and arrows as before that has the almost amalgamation property. The following conditions are equivalent:

(a) *$\mathcal{C}$ is separable.*

(b) *$\mathcal{C}$ has a Fraïssé sequence.*

Consequences

Theorem (Kubiś)

Under assumptions of the previous theorem and separability we have:

1. *Uniqueness* There exists exactly one Fraïssé sequence $\vec{U}$ (up to an isomorphism).
2. *Universality* For every sequence $\vec{X}$ in $\mathcal{C}$ there is an arrow $f\colon \vec{X} \to \vec{U}$.
3. *Almost homogeneity* For every $A, B \in \mathrm{Ob}(\mathcal{C})$ and for all arrows $i\colon A \to \vec{U}$, $j\colon B \to \vec{U}$, for every arrow $f\colon A \to B$, for every $\varepsilon > 0$, there exists an isomorphism $H\colon \vec{U} \to \vec{U}$ such that $d(j \circ f, H \circ i) < \varepsilon$.

In our examples we will have almost homogeneity for sequences in $\mathcal{C}$ as well.

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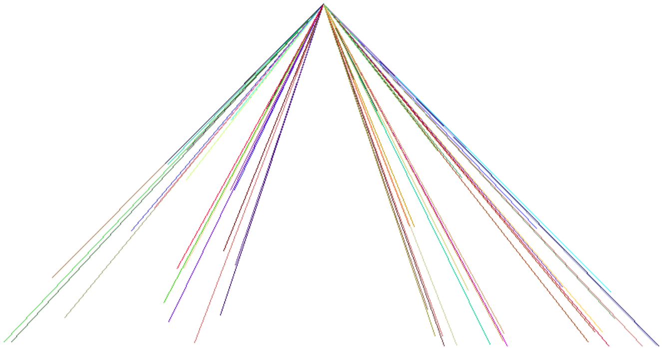
- Lelek fan $\mathbb{L}$ is a non-trivial closed connected subset of V containing the top point, which has a dense set of endpoints in $\mathbb{L}$

Lelek fan



About the Lelek fan

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- Lelek fan is unique: any two are homeomorphic (Bula-Oversteegen 1990 and Charatonik 1989)

Geometric fans

Definition

A geometric fan is a closed connected subset of the Cantor fan containing the top point

The category

The category $\mathfrak{F}$

- Objects are finite geometric fans, metric inherited from $\mathbb{R}^2$.

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- $f\colon F \to G$ is a stable embedding if it is a one-to-one affine map such that endpoints are mapped to endpoints.

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- An arrow from F to G is a pair $\langle e, p\rangle$ such that $e\colon F \to G$ is a stable embedding, $p\colon G \to F$ is a 1-Lipschitz affine surjection and $p \circ e = \mathrm{id}_F$.

Properties

- Geometric fans = inverse limits of sequences in $\mathfrak{F}$
- The category $\mathfrak{F}$ is directed and has the strict amalgamation property
- $\mathfrak{F}$ is a separable metric category

Fraïssé sequences

Theorem (Kubiś - K)

Let $\vec{U}$ be a sequence in $\mathfrak{F}$ and let U_∞ be its inverse limit. The following properties are equivalent:

(a) *The set of endpoints $E(U_\infty)$ is dense in U_∞.*

(b) *$\vec{U}$ is a Fraïssé sequence.*

Consequences

- uniqueness of a Fraïssé sequence
  The Lelek fan is a unique smooth fan whose set of end-points is dense.
- universality with respect to all geometric fans
  For every geometric fan F there are a stable embedding e into the Lelek fan $\mathbb{L}$ and a 1-Lipschitz affine retraction p from $\mathbb{L}$ onto F such that $p \circ e = \mathrm{id}_F$.

Consequences

- almost homogeneity with respect to all geometric fans
  Let F be a geometric fan stably embedded in $\mathbb{L}$ and let $f, g \colon \mathbb{L} \to F$ be continuous affine surjections. Then for every $\varepsilon > 0$ there is a homeomorphism $h \colon \mathbb{L} \to \mathbb{L}$ such that for every $x \in \mathbb{L}$, $d_F(f \circ h(x), g(x)) < \varepsilon$.

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Remark

in 2015, Bartošová and Kwiatkowska obtained uniqueness, universality, and almost homogeneity of the Lelek fan in the context of the projective Fraïssé theory.

Extreme points

Definition

A point x in a compact convex set K of a topological vector space is an extreme point if whenever $x = \lambda y + (1 - \lambda)z$ for some $\lambda \in [0, 1]$, $y, z \in K$, then $\lambda = 0$ or $\lambda = 1$.
The set of extreme points of K is denoted by ext K.

Simplices

Definition

A simplex is a non-empty compact convex and metrizable set K in a locally convex linear topological space such that every $x \in K$ has a unique probability measure μ supported on ext K and such that

$$f(x) = \int_K f \, d\mu$$

for every continuous affine function $f \colon K \to \mathbb{R}$.

Finite dimensional simplices

Example

Finite-dimensional simplex Δ_n

$$\{x \in \mathbb{R}^{n+1} \colon \sum_{i=1}^{n+1} x(i) = 1 \text{ and } x(i) \geq 0 \text{ for every } i = 1, \ldots, n+1\}$$

In particular, Δ_0 is a singleton, Δ_1 is a closed interval, and Δ_2 is a triangle.

The Poulsen simplex

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Uniqueness was proved by Lindenstrauss, Olsen, and Sternfeld in '78.

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- $p\colon L \to K$ is affine if for any $x, y \in L$ and $\lambda \in [0,1]$ we have $p(\lambda x + (1-\lambda)y) = \lambda p(x) + (1-\lambda)p(y)$.
- Stable embedding is a one-to-one affine map such that extreme points are mapped to extreme points.

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- Stable embedding is a one-to-one affine map such that extreme points are mapped to extreme points.
- An arrow from K to L is a pair $\langle e, p\rangle$ such that $e\colon K \to L$ is a stable embedding, $p\colon L \to K$ is an affine projection and $p \circ e = \mathrm{id}_K$.

Properties

Theorem (Lazar-Lindenstrauss '71)

Metrizable simplices are, up to affine homeomorphisms, precisely the limits of inverse sequences in $\mathfrak{S}$.

- The category $\mathfrak{S}$ is directed and has the strict amalgamation property
- $\mathfrak{S}$ is a separable metric category

Fraïssé sequences

Theorem (Kubiś - K)

Let $\vec{U}$ be a sequence in $\mathfrak{S}$ and let K be its inverse limit. The following properties are equivalent:

(a) *The set* ext K *is dense in* K.

(b) $\vec{U}$ *is a Fraïssé sequence.*

Consequences

- uniqueness of a Fraïssé sequence
  The Poulsen simplex $\mathbb{P}$ is unique, up to affine homeomorphisms.
- universality with respect to all simplices
  Every metrizable simplex is affinely homeomorphic to a face of $\mathbb{P}$.

Consequences

- almost homogeneity with respect to all simplices
  Let F be a simplex and let $f, g\colon \mathbb{P} \to F$ be affine and continuous. Then for every $\varepsilon > 0$ there is an affine homeomorphism $H\colon \mathbb{P} \to \mathbb{P}$ such that for every $x \in \mathbb{P}$, $d_F(f \circ H(x), g(x)) < \varepsilon$, where d_F is a fixed compatible metric on F.

Remark

Uniqueness, universality, and homogeneity of $\mathbb{P}$ were proved by Lindenstrauss, Olsen, and Sternfeld in '78.

Homogeneity results

Remark

Let $S, T \subseteq E(\mathbb{L})$ be finite sets. Then there exists an affine homeomorphism $h\colon \mathbb{L} \to \mathbb{L}$ such that $h[S] = T$

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Theorem (Kubiś - K)

Let $A, B \subseteq E(\mathbb{L})$ *be countable dense sets. Then there exists an affine homeomorphism* $h\colon \mathbb{L} \to \mathbb{L}$ *such that* $h[A] = B$.

Comments

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- There exists a homeomorphism $h\colon E(\mathbb{L}) \to E(\mathbb{L})$ such that for no homeomorphism $f\colon \mathbb{L} \to \mathbb{L}$, we have $f \restriction E(\mathbb{L}) = h$.

Generalization of the category $\mathfrak{F}$

- F be a geometric fan
- $E(F)$ - the set of endpoints of F
- A skeleton in F is a convex set $D \subseteq F$ such that $E(D)$ is countable, contained in $E(F)$ and dense in $E(F)$.

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- Let $\mathfrak{F}^d$ be the category whose objects are pairs of finite geometric fans (F^1, F^2) with $F^1 = F^2$.

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- An arrow from (F^1, F^2) to (G^1, G^2) is a pair $\langle e, p \rangle$ such that $e \colon F^1 \to G^1$ is a stable embedding, $p \colon G^2 \to F^2$ is a 1-Lipschitz affine retraction and $p \circ e = \mathrm{id}_F$.

Generalization of the category $\mathfrak{F}$

- The category $\mathfrak{F}^d$ is directed and has the strict amalgamation property.
- $\mathfrak{F}^d$ is a separable metric category, therefore it has a unique up to isomorphism Fraïssé sequence.
- Its limit is $(D, \mathbb{L})$ for some skeleton D in $\mathbb{L}$.

Generalization of the category $\mathfrak{F}$

To show the main theorem we need the following lemma:

Lemma

Let L be a geometric fan and let D be a skeleton in L. Then there exist a geometric fan L', a skeleton D' of L', and an affine (not necessarily 1-Lipschitz) homeomorphism $h\colon L \to L'$ with $h(D) = D'$ such that there is a sequence $\vec{F}$ in $\mathfrak{F}^d$ satisfying $L' = \varprojlim \vec{F}$ and $D' = \varinjlim \vec{F}$.