Squares of function spaces and function spaces on squares

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If X is metrizable and $A \subseteq X$ is closed, then
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It follows that, e.g.

$$C_p([0,1]) \approx C_p([0,1]) \times C_p([0,1])$$
$$C_p(\mathbb{R}) \approx C_p(\mathbb{R}) \times C_p(\mathbb{R})$$

Problem (Arhangel'skii), 1978, 1990

Is it true that $C_p(X)$ is homeomorphic to $C_p(X) \times C_p(X)$ provided X is an infinite 'nice' topological space, e.g. is compact or metrizable?

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  - Factorization properties help constructing homeomorphisms between function spaces.
- Related to another important question: Which topological properties of $C_p(X)$ are productive?
  - **Open question:** Suppose that $C_p(X)$ is Lindelöf. Is it true that $C_p(X) \times C_p(X)$ is Lindelöf?

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Marciszewski example

$X = \omega \cup \{p_A : A \in \mathcal{A}\} \cup \{\infty\}$, where $\mathcal{A}$ is a suitable almost disjoint family on ω. Points in ω are isolated, neighborhoods of p_A are of the form $\{p_A\} \cup (A \setminus F)$, where F is finite.

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A nontrivial metrizable continuum M is a *Cook continuum* if it is rigid, i.e. for any subcontinuum $C \subseteq M$, each continuous function $f : C \to M$ is either the identity or $f = const$.

Theorem (K. & Marciszewski, 2015)

There is an infinite zero-dimensional subspace B of the real line (a rigid Bernstein set), such that $C_p(B)$ is not homeomorphic to $C_p(B) \times C_p(B)$.

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The rigid Bernstein set B

- Let $\{(C_\alpha, f_\alpha) : \alpha < 2^\omega\}$ be the collection of all pairs (C, f), where C is a copy of the Cantor set in $\mathbb{R}$ and $f : C \to \mathbb{R}$ is a continuous map with uncountable range $f(C)$ disjoint from C.

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B is rigid in the following sense: If G is an uncountable G_δ-subset of B, then for each continuous function $f : G \to B$ there exists an uncountable G_δ-subset G' of G such that the restriction $f \upharpoonright G'$ is either the identity or is constant.

Theorem (Marciszewski, 2000)

Suppose that X and Y are metrizable. Let $n \in \mathbb{N}$ and suppose that $\Psi : C_p(X) \to C_p(Y)$ is a homeomorphism with $\Psi(\underline{0}) = \underline{0}$.

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For every $r \in \mathbb{N}$ there are continuous maps $f_1^r, \ldots, f_{p_r}^r : G_r \to X$ and $m \in \mathbb{N}$ such that, for any $y \in G_r$, $\Psi(O_X(A, \frac{1}{m})) \subseteq \overline{O}_Y(y, \frac{1}{n})$, where $A = \{f_1^r(y), \ldots, f_{p_r}^r(y)\}$.

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- We can identify $C_p(B) \times C_p(B)$ with $C_p(B \oplus B)$
- Using rigidity of B we can conclude that the mapping in the above theorem, restricted to an uncountable G_δ, are either the identity or are constant

- Define a mapping $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
  $\varphi(t_1, t_2) = \Phi^{-1}(t_1 v_1 + t_2 v_2)(c)$, where $v_1, v_2 \in C_p(B \oplus B)$ and $c \in B$ are suitably chosen.

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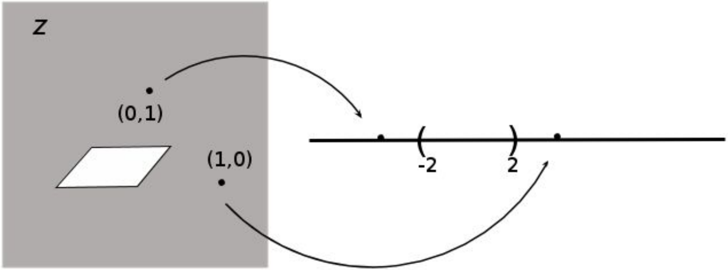

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A natural candidate for a counterexample is the Cook continuum M used in the context of linear and uniform homeomorphisms.

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'Yes' if X is either non-scattered or is scattered of height $\leq \omega$ (Baars, de Groot, 1992).

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If $X = M$ or $X = B$, then there is no linear continuous surjection of $C_p(X)$ onto $C_p(X) \times C_p(X)$.

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Corollary (K. & Marciszewski)

No, If M is a Cook continuum, then there is no linear continuous surjection of $C_p(M)$ onto $C_p(M \times M)$.

Question (Kawamura & Leiderman, 2016)

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A continuum X is *pseudo-rigid* if for any continuum C and continuous map $F : X \times C \to X$ we have
$(\forall c \in C)\ F \restriction X \times \{c\} = F \restriction X \times \{c_0\}$, for some $c_0 \in C$ or
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Question (Łysko, 2007)

Let $r : P \times P \to \Delta = \{(x, y) \in P \times P : x = y\}$ be a continuous retraction. Must r be of the form $r(x, y) = (x, x)$ or $r(x, y) = (y, y)$ for all $(x, y) \in P \times P$?

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As a set, $A(X)$ consists of elements of the form $\sum_{i=1}^{n} a_i x_i$, where $a_i \in \mathbb{Z}$, $x_i \in X$ and $n \in \mathbb{N}$.

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Theorem (K. & Leiderman, 2016)

If M is a Cook continuum, then $A(M \times M)$ does not embed into $A(M)$ as a subgroup.

Thank you!