Homogeneous spaces as coset spaces of groups from special classes

K. Kozlov
Lomonosov Moscow State University

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Contents

Example I

In the study of topological homogeneity it is natural to ask from what class of groups we can choose a group that realizes one or the other kind of space's homogeneity.

Example I. How the knowledge about a group which realizes homogeneity allows to deduce stronger homogeneity properties of a space from weaker one

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From these results G. Ungar [1975] deduced that a metrizable homogeneous compactum (even a homogeneous separable metrizable locally compact space) is a coset space of a Polish group.

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For a topological group G and its closed subgroup H the left *coset space* G/H is a G-space $(G/H, G, \alpha)$ with the action of G by left translations $\alpha : G \times G/H \to G/H$, $\alpha(g, hH) = ghH$.

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COSET SPACES $\subset$ HOMOGENEOUS SPACES

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This question has a positive answer in the case of strongly locally homogeneous spaces.

SLH spaces

Definition (L. Ford 1954)

A space X is *strongly locally homogeneous* (abbreviated, SLH) if it has an open base $\mathbb{B}$ such that for every $B \in \mathbb{B}$ and any $x, y \in B$ there is a homeomorphism $f : X \to X$ which is supported on B (that is, f is identity outside B) and moves x to y.

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K. Kozlov [2013] showed that any separable metrizable SLH space has an extension that is a Polish SLH space.

R. Arens [1946]: $\mathrm{Hom(X)}$ is a topological group which action on X is continuous.

If X is a compactum (even a locally compact space) then we have

$$\mathrm{w(Hom(X))} \leq \mathrm{w(X)}.$$

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Question 2. Is a coset space X a coset space of some group G with $\mathrm{w}(G) \leq \mathrm{w}(X)$?

Partial answer on Questions 1, 2

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Definition (F. Ancel 1986, K. Kozlov, V. Chatyrko 2010)

The action $\alpha : G \times X \to X$ is called

- *open (or micro-transitive)* if $x \in \mathrm{Int}(Ox)$ for any point $x \in X$ and any nbd $O \in N_G(e)$;
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The sets $\mathrm{Int}\, A$, $\mathrm{Cl}\, A$ are the interior and closure of a subset A, respectively, $N_G(e)$ denotes the family of open neighborhoods of the unit e of a group G, $Ox = \bigcup\{gx : g \in O\}$ for $O \in N_G(e)$, $x \in X$.

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If (X, G, α) is a G-space with a d-open action, then X is a direct sum of clopen subsets (*components of the action*). Each component of the action is the closure of the orbit of an arbitrary point of this component.

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Partial answer on Questions 1, 2

Theorem

For a G-space (X, G, α) with a d-open action there exist

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III. H is H' in compact-open topology.

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Corollary

If (X, G, α) is a G-space with a d-open action and X is a separable metrizable space then there exist

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Questions. When a separable metrizable coset space is a coset space of a separable metrizable group?

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When a (separable metrizable) coset space has a (metrizable) compactification which is a coset space?

(d-)open or (weakly) micro-transitive actions

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Corollary

If a compactum X is a space with a d-open action of an ω-balanced or a Čech complete group then X is a Dugundji compactum.

Decompositions of actions

Coset spaces of compact metrizable groups

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A space X is *metrically homogeneous* if there is a compatible metric on X such that its group of isometries acts transitively on X.

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Necessity follows from the result of L. Kristensen [1958] and sufficiency from the result of R. Arens [1946].

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S. Antonyan, T. Dobrowolski [2015], K. H. Hofmann, L. Kramer, [2015]. Hilbert cube is an example of a coset space which is not a coset space of a compact group.

Decomposition of actions

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A pair of maps $(f : X \to Y,\ \varphi : G \to H)$ of (X, G, α_G) to (Y, H, α_H) such that $\varphi : G \to H$ is a homomorphism and the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi \times f} & H \times Y \\ \downarrow \alpha_G & & \downarrow \alpha_H \\ X & \xrightarrow{f} & Y \end{array}$$

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By a *separable metrizable G-space* (respectively compact metrizable G-space) we understand a G-space (X, G, α) where X and G are separable metrizable (respectively compact metrizable) spaces.

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Theorem

A compactum X is a coset space of a compact group iff there is $G \in \mathcal{D}$ and a family of equivariant maps $(f_\gamma, \varphi_\gamma)$ of (X, G, α) to compact metrizable G-spaces $(X_\gamma, G_\gamma, \alpha_\gamma)$, $\gamma \in \mathcal{A}$, such that the family of maps f_γ, $\gamma \in \mathcal{A}$, on X is separating.

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In fact this theorem is a reformulation of Okromeshko's theorem.

Decomposition of actions

Theorem

X is a (d-)coset space of an ω-narrow group iff there is $G \in \mathcal{O}D$ ($\mathcal{D}$) and a family of equivariant maps $(f_\gamma, \varphi_\gamma)$ of (X, G, α) to separable metrizable G-spaces $(X_\gamma, G_\gamma, \alpha_\gamma)$ with (d-) open actions α_γ, $\gamma \in \mathcal{A}$, such that the family of maps f_γ, $\gamma \in \mathcal{A}$, on X is separating.

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Let $(\mathrm{id}, \varphi) : (X, G, \alpha_G) \to (X, H, \alpha_H)$ be an equvariant pair of maps, where $H = \varphi(G)$. Then if the action α_G is (d-)open then the action α_H is (d-)open respectively.

Decomposition of actions

Theorem

Let (X, G, α) be a G-space with an (d-) open action and let H be the kernel of an epimorphism $\varphi : G \to G'$. Then for the pseudouniformity $\mathcal{U}_{G'}$ on X which base consists of covers

$$\gamma_O = \{\mathrm{Int}((\varphi^{-1}O)x) : x \in X\}, \ O \in N_{G'}(e),$$

we have:

(a) *(π, φ) is an equivariant pair of maps, where $\pi : X \to X/\mathcal{U}_{G'}$ is a uniform quotient map of X on a uniform quotient space $X/\mathcal{U}_{G'}$;*

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If $\mathcal{U}$ is a pseudouniformity on X then the subsets $[x]_{\mathcal{U}} = \bigcap\{\mathrm{St}(x, \upsilon) : \upsilon \in \mathcal{U}\}$ form a partition $E(\mathcal{U})$ of X. On the quotient set $X/E(\mathcal{U})$ with respect to this partition the *quotient uniformity* $\bar{\mathcal{U}}$ is defined. It is the greatest uniformity on $X/E(\mathcal{U})$ such that the quotient map $p : X \to X/E(\mathcal{U})$ is uniformly continuous.

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Decomposition of actions

Corollary

For a pseudocompact space X the following conditions are equivalent:

(a) *X is a (d-)coset space of an ω-narrow group;*

(b) *X is a (d-)coset space of an ω-balanced group;*

(c) *X is an $\mathbb{R}$-factorizable G-space for some $G \in \mathcal{D}$ $(\mathcal{OD})$;*

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A G-space (X, G, α) is said to be $\mathbb{R}$-*factorizable*, if for every continuous real-valued function f on X there exist a separable metrizable G-space (Y, H, α_H), an equivariant pair of maps $(g : X \to Y,\ \varphi : G \to H)$ and a map $h : Y \to \mathbb{R}$ such that $f = h \circ g$.

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Theorem (E. Martyanov 2016)

A compact coset space X is a coset space of an ω-narrow group iff (X, G, α) is $\mathbb{R}$-factorizable for some $G \in \mathcal{O}D$.