Tame Locally Convex Spaces

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"Such a considerable flourish of examples [James's and Tsirelson's spaces] had at least one consequence: everyone got lost. Nobody knew any longer what to expect, and even the most impetuous newcomers could hardly make any conjecture, which, for a mathematician, is a sad situation. ***The only general structure theorem which has been proved since then was Rosenthal's, dealing with*** l^1 ***and weak Cauchy subsequence****."*

Bernard Beauzamy, 1997, [1]

Gratitude

All of the results presented here are from a joint work with Michael Megrelishvili:

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We are indebted to Saak Gabriyelyan and Arkady Leiderman for their important suggestions.

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- A generalized Haydon's theorem.
- Extension of a result of Ruess (2014) about Rosenthal's dichotomy.
- A general framework for "smallness" of locally convex spaces.

Tame Families

Definition

A sequence of real functions $\{f_n \colon X \to \mathbb{R}\}_{n \in \mathbb{N}}$ on a set X is said to be **independent** *(Rosenthal 1974) if there exist real numbers $a < b$ such that*

$$\bigcap_{n \in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n \in M} f_n^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets P, M of $\mathbb{N}$.

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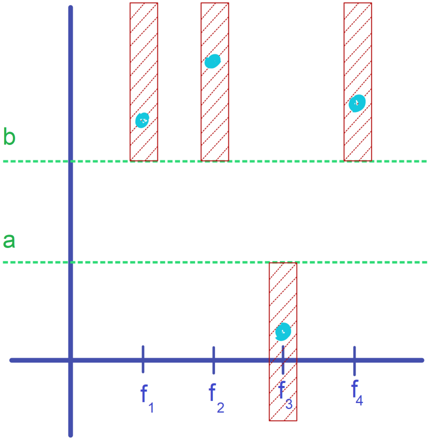
Exercise

If $\{f_n\}_{n\in\mathbb{N}}$ *is* **not** *tame over* X*, then* X *is not tame over* $\{f_n\}_{n\in\mathbb{N}}$*.*

Tame Families

Visualization

Figure: here $P = \{3\}$ and $M = \{1, 2, 4\}$



Fragmented Functions

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- *A real-valued function f on a topological space (X, τ) is said to be* **fragmented** *if for every subset $A \subseteq X$ and $\varepsilon > 0$, there exists an open $O \in \tau$ such $A \cap O \neq \emptyset$ and $\operatorname{diam} f(A \cap O) < \varepsilon$.*

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- *A bounded family $F \subseteq \mathbb{R}^X$ is said to be* **eventually fragmented** *if every sequence in F contains a fragmented subsequence.*

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These are all well-known from Rosenthal. One can also show that the following are also equivalent to them:

- *The ball B_V is tame (equivalently, eventually fragmented) over B_{V^*}.*
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Tame families have been useful in the study of representations of dynamical systems on Rosenthal Banach spaces in several joint papers of Glasner and Megrelishvili and also in a paper about tame functionals on Banach algebras.

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Fact (Glasner-Megrelishvili)

A compact space X is WRN (i.e., embedded into the dual of a Rosenthal Banach space with its weak-star topology) iff there exists a tame family F of continuous functions on X which separates the points of X.

Tame Locally Convex Spaces

Definition (New)

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Proposition

A Banach space is a tame lcs iff it is a Rosenthal Banach space.

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Every lcs with a separable dual is NP.

Stability Properties of Tame and NP Locally Convex Spaces

Theorem

The classes $(\mathbf{T})$ *and* $(\mathbf{NP})$ *are closed under taking:*

1. *subspaces*
2. *bound covering maps*
3. *products*
4. *direct sums*

Moreover, if F *is a large, dense subspace of the locally convex space* E*, and* $F \in (\mathbf{T})$ *(resp.* $F \in (\mathbf{NP})$*), then* $E \in (\mathbf{T})$ *(resp.* $E \in (\mathbf{NP})$*). In particular, if* V *is a normed tame (NP) space, then so is its completion.*

Examples

- If the dual V^* is a linear subspace in a product of separable lcs, then V is (**NP**).
- There exists an NP (even DLP) space which can't be embedded in the product of Rosenthal Banach spaces.
- A compact G-system is representable on a tame lcs if and only if it is tame.

Haydon's Theorem

Fact (important)

Haydon, 1976 *Let V be a Banach space. The following are equivalent:*

1. *V contains no l^1-sequence (i.e. V is a Rosenthal Banach space);*
2. *every weak-star compact convex subset of V^* is the norm closed convex hull of its extreme points;*
3. *for every weak-star compact subset T of V^*,*

$$\overline{\mathrm{co}}^{w^*}(T) = \overline{\mathrm{co}}(T).$$

Generalized Haydon's Theorem

One of Our Main Results

Theorem

For a locally convex space E, the following are equivalent:

1. *E is tame (equivalently, does not contain an l^1-sequence);*
2. *every equicontinuous, weak-star compact convex subset of E^* is the strong closed convex hull of its extreme points. That is, $\overline{\mathrm{co}}^{w^*}(\mathrm{ext}\, M) = \overline{\mathrm{co}}(\mathrm{ext}\, M)$ for every convex $M \in \mathrm{eqc}\,(E^*)$;*
3. *for every equicontinuous, weak-star compact subset T of E^*,*

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What About l^1-Sequences?

Definition

Let E be a locally convex space. A bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ is said to be **a generalized l^1-*sequence*** *if there exist: a continuous seminorm ρ on E and $\delta > 0$, such that for every $c_1, \dots, c_n \in \mathbb{R}$*

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A locally convex space E is tame if and only if it has no bounded l^1-sequence. If E is locally complete, then it is equivalent to l^1 not being embedded inside it.

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$(\mathbf{Ros}) \subseteq (\mathbf{T})$.

Theorem

There exists a tame complete (even reflexive) lcs which:

(i) *is not a Rosenthal lcs;*

(ii) *does not contain any l^1-subsequence;*

(iii) *contains a dense, Rosenthal subspace.*

As a corollary: Rosenthal's dichotomy does not hold for such locally convex spaces.

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Example

$\mathbb{R}^{[0,1]}$.

The Case of Spaces With Metrizable Bounded Subsets

Extension of a Result of Ruess, 2014

Theorem

If E is a lcs with metrizable bounded subsets, then the following are equivalent:

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- *E has no bounded l^1-sequences.*
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This extends a known result due to W.M. Ruess (2014) about locally complete spaces with metrizable bounded subsets.

DLP Spaces

Definition

Let $F \subset \mathbb{R}^K$ be a family of real functions on a set K. Then F is said to have the **double limit property (DLP)** *if for every sequence $\{f_n\}_{n\in\mathbb{N}}$ in F and every sequence $\{x_n\}_{n\in\mathbb{N}}$ in K, the limits*

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Examples of DLP spaces

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Example

The following are (**DLP**):

- Semi-reflexive lcs;
- Schwartz lcs (as a subspace of a reflexive lcs);
- Quasi-Montel lcs
- For every locally convex space E, the lcs (E, w) with its weak topology is (**DLP**).
- Every space $C_p(X)$, in its pointwise topology (for every topological space X), is (**DLP**).

Example: Free Locally Convex Spaces and the DLP Bornological Class

Fact (Leiderman and Uspenskij, 2021)

- *$L(K)$ is multi-reflexive for every compact K.*
- *The space $L(P)$ where $P = \mathbb{N}^{\mathbb{N}}$ is the space of irrational numbers is* **not** *multi-reflexive.*

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Theorem

Let X be a Dieudonné complete space. Then $L(X)$ is semi-reflexive if and only if X has no infinite compact subset.

The Case of $C(X)$ for Scattered X

Fact (Pełczyński and Semadeni, 1959)

Let K be a compact space. The following are equivalent:

1. *l^1 cannot be embedded in $C(K)$.*
2. *The dual of every separable Banach subspace of $C(K)$ is separable.*
3. *K is scattered.*

In 2015, Gabriyelyan–Kakol–Kubiś–Marciszewski gave a natural generalization for the lcs $C_k(X)$ where X is a Tychonoff space.

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Fact (GKKM15)

For every Tychonoff space X the following are equivalent:

1. *$C_k(X)$ contains a copy of l^1.*
2. *$C_k(X)$ contains a separable Banach space V with non-separable dual.*
3. *X contains a non scattered compact set.*

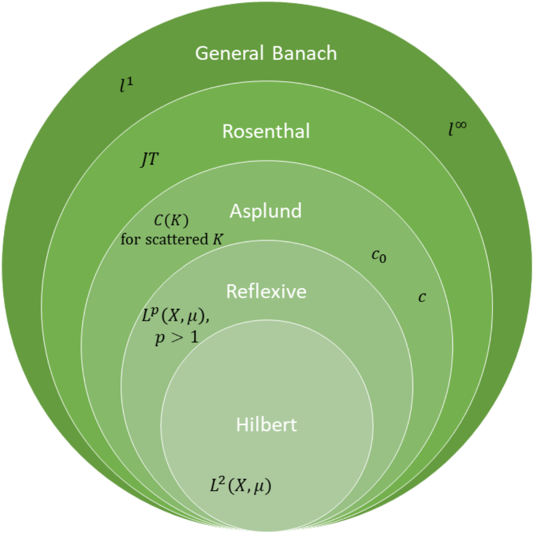
The Case of $C(X)$ for Scattered X

Proposition (New)

For every Tychonoff space X the following are equivalent:

1. $C_k(X)$ *is a tame lcs.*
2. $C_k(X)$ *is* $(\mathbf{NP})$*.*
3. *Every compact subset of* X *is scattered.*

A Map of Banach Spaces



Possible Generalization Methods

Reflexive	Asplund	Rosenthal
The evaluation map $J\colon V \to V^{**}$ is an isomorphism	Every separable subspace of V has a separable dual	l^1 is not embedded in V
The closed unit ball of V is weak compact	Frechet differentiability on dense G_δ subsets	Every bounded sequence has a weak-Cauchy subsequence
Every bounded sequence in V has a weakly convergent subsequence	every non-empty bounded subset of V^* has weak*-slices of arbitrarily small diameter	Every element of V^{**} is the weak-star limit of elements of V
B_V has the DLP viewed as a family of functions over B_{V^*}	B_V is fragmented viewed as a family of functions over B_{V^*}	B_V is tame viewed as a family of functions over B_{V^*}

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What Makes a Family of Functions Small?

Definition

A **bornological class** $\mathfrak{B}$ *is an assignment*

$$Comp \to \{Bornologies\}, \quad K \mapsto \mathfrak{B}_K$$

from the class of all compact spaces Comp to the class of vector bornologies such that $\mathfrak{B}_K$ *is a separated convex vector bornology on the Banach space* $C(K)$ *satisfying the following properties:*

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2. ***consistency***: *Suppose that* $\varphi\colon K_1 \to K_2$ *is a continuous map.*
   2.1 *If* $A \in \mathfrak{B}_{K_2}$, *then* $A \circ \varphi \in \mathfrak{B}_{K_1}$.
   2.2 *If* φ *is surjective, then the converse is also true, namely that* $A \circ \varphi \in \mathfrak{B}_{K_1}$ *implies* $A \in \mathfrak{B}_{K_2}$.

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3. ***Bipolarity****: If* $A \in \mathfrak{B}_K$*, then* $A^{\circ\circ} = \overline{\mathrm{acx}}^{w} A \in \mathfrak{B}_K$ *where the polar is taken with respect to the dual* $C(K)^*$ *(note that we use the Bipolar Theorem).*

Examples of Bornological Classes

- [**DLP**] - Bounded families satisfying the DLP.
- [**NP**] - Fragmented families.
- [**T**] - Tame/eventually fragmented families.

Back to Locally Convex Spaces

Definition

*Let $\mathfrak{B}$ be a bornological class. A bounded subset $B \subseteq E$ is said to be $\mathfrak{B}$-**small** if for every $M \in \mathrm{eqc}\,(E^*)$, $r_M(B) \in \mathfrak{B}_M$ where $r_M \colon E \to C(M)$ is the restriction operator.*
*A locally convex space is said to be $\mathfrak{B}$-**small** if every bounded subset is $\mathfrak{B}$-small.*

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Lemma

Let E be a locally convex space and $\mathfrak{B}$ a bornological class. The family of $\mathfrak{B}$-small subsets in E is a saturated, convex vector bornology, denoted by $\operatorname{small}(\mathfrak{B}, E)$.

Properties of $\mathfrak{B}$-Small Spaces

Theorem

The class of $\mathfrak{B}$-small locally convex spaces is closed under:

1. *subspaces*
2. *bound covering maps*
3. *products*
4. *direct sums*
5. *inverse limits.*

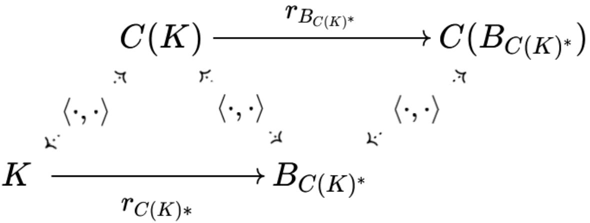
Moreover, if F is a large, dense subspace of the locally convex space E, and F is $\mathfrak{B}$-small, then so is E. In particular, if V is a normed $\mathfrak{B}$-small space, then so is its completion.

Co-$\mathfrak{B}$-Small Subsets

Definition

A bornological class $\mathfrak{B}$ is said to be **polarly compatible** *if whenever $A \in \mathfrak{B}_K$ for compact K, then $r_{B_{C(K)^*}}(A) \in \mathfrak{B}_{B_{C(K)^*}}$ where $r_{B_{C(K)^*}} : C(K) \to C(B_{C(K)^*})$ is the canonical map defined by:*

$$(r_{B_{C(K)^*}}(f))(\varphi) := \varphi(f).$$



Co-$\mathfrak{B}$-Small Subsets

Definition

Let $\mathfrak{B}$ be a bornological class and $A \subseteq E$ is a bounded subset. An equicontinuous, $M \subseteq E^*$ is said to be **co-$\mathfrak{B}$-small with respect to** *A if $r(A) \in \mathfrak{B}_{\overline{M}^{w*}}$, where $r\colon E \to C(\overline{M}^{w^*})$ is the restriction map. If this is true for every bounded subset of E, then we will simply say that M is* **co-$\mathfrak{B}$-small**.

Lemma

Let $\mathfrak{B}$ be a polarly compatible bornological class and let $A \subseteq E$ be bounded. The family of co-$\mathfrak{B}$-small subsets with respect to A of E^* is a **weak-star saturated, convex bornology**. *Denote this bornology as $\mathrm{small}^*(\mathfrak{B}, E, A)$. We also write*

$$\mathrm{small}^*(\mathfrak{B}, E) := \bigcap_{A \subseteq E} \mathrm{small}^*(\mathfrak{B}, E, A)$$

where A runs over bounded subsets. Clearly, $\mathrm{small}^*(\mathfrak{B}, E)$ is also a weak-star saturated, locally convex bornology.

Strongest $\mathfrak{B}$-Small topology

Definition

Let $\mathfrak{B}$ be a polarly compatible bornological class, and (E, τ) be a locally convex space. Recall that Lemma -1.37 applies in this case so $\mathrm{small}^*(\mathfrak{B}, E)$ is a convex bornology. We define $\tau_{\mathfrak{B}}$ to be the polar topology generated by $\mathrm{small}^*(\mathfrak{B}, E)$. Since $\mathrm{small}^*(\mathfrak{B}, E)$ consists of equicontinuous subsets, $\tau_{\mathfrak{B}} \subseteq \tau$.

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Theorem

For every lcs (E, τ), $\tau_{\mathfrak{B}}$ is the strongest locally convex, $\mathfrak{B}$-small topology coarser than τ.

More Applications of Bornological Classes

- A generalization of co-tame subsets.
- Defining the strongest $\mathfrak{B}$-small topology on a given space.
- Relation to the Mackey topology and other similar definitions.

The DJFP Factorization

Davis-Figiel-Johnson-Pelczyński

Definition

We say that a linear continuous map $T: E \to F$ *between lcs is* **tame** *if there exists a zero neighborhood* $U \subseteq E$ *such that* $T(U) \subseteq F$ *is a tame subset in* F*.*

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Proposition

Every tame operator $T: E \to X$ *between a lcs* E *and a Banach space* X *can be factored through a Rosenthal Banach space.*

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