Random elements of large groups – Continuous case

Viktor Kiss

Eötvös Loránd University

Toposym,
July 25, 2016

Joint work with Udayan B. Darji, Márton Elekes, Kende Kalina, Zoltán Vidnyánszky

Random elements of large groups

The main question of the talk somewhat vaguely is the following:

Question

How does the random element of large topological groups behave?

Random elements of large groups

The main question of the talk somewhat vaguely is the following:

Question

How does the random element of large topological groups behave?

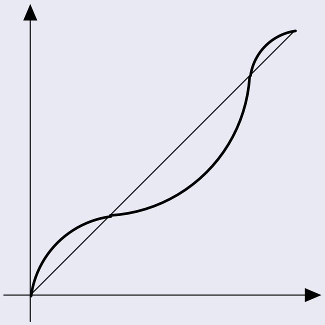
Example

In S_∞, the permutation group of the countably infinite set, two elements behave similarly if they have the same the cycle decomposition.

Random elements of large groups

Example

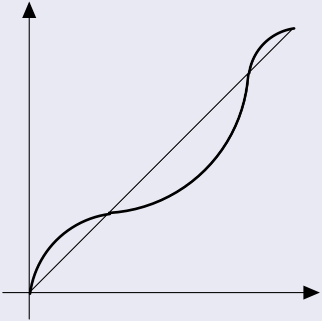
In $\text{Homeo}^+([0,1])$ two elements $f, g \in \text{Homeo}^+([0,1])$ behave similarly, if there is a homeomorphism $\psi \in \text{Homeo}^+([0,1])$ such that $f(\psi(x)) > \psi(x)$, $f(\psi(x)) < \psi(x)$ and $f(\psi(x)) = \psi(x)$ iff $g(x) > x$, $g(x) < x$ and $g(x) = x$, respectively.

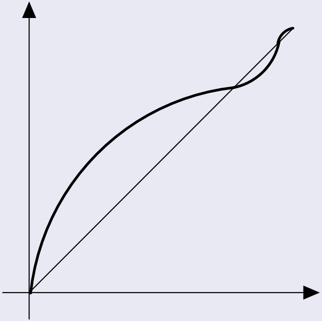


Random elements of large groups

Example

In $\text{Homeo}^+([0,1])$ two elements $f, g \in \text{Homeo}^+([0,1])$ behave similarly, if there is a homeomorphism $\psi \in \text{Homeo}^+([0,1])$ such that $f(\psi(x)) > \psi(x)$, $f(\psi(x)) < \psi(x)$ and $f(\psi(x)) = \psi(x)$ iff $g(x) > x$, $g(x) < x$ and $g(x) = x$, respectively.





In both cases, conjugacy describes the similar behavior, hence we deal with the size of conjugacy classes.

Haar null sets

We only deal with Polish groups, that is, the topology is separable and completely metrizable.

Haar null sets

We only deal with Polish groups, that is, the topology is separable and completely metrizable.

Definition (Christensen)

Let G be a Polish topological group. A subset $H \subset G$ is called *Haar null* if there is exists a Borel set $B \supset H$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.

Haar null sets

We only deal with Polish groups, that is, the topology is separable and completely metrizable.

Definition (Christensen)

Let G be a Polish topological group. A subset $H \subset G$ is called *Haar null* if there is exists a Borel set $B \supset H$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.

Theorem (Christensen)

The family of Haar null sets form a σ-ideal.

Haar null sets

We only deal with Polish groups, that is, the topology is separable and completely metrizable.

Definition (Christensen)

Let G be a Polish topological group. A subset $H \subset G$ is called *Haar null* if there is exists a Borel set $B \supset H$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.

Theorem (Christensen)

The family of Haar null sets form a σ-ideal.
If G is locally compact then $H \subset G$ is Haar null if and only if H is of measure zero with respect to a left (or equivalently, a right) Haar measure defined on G.

Previous results concerning Haar null sets

Theorem (Christensen)

Let X be a separable Banach space and $f : X \to \mathbb{R}$ a Lipschitz function. Then f is Gâteaux differentiable almost everywhere (that is, the set of those points $x \in X$ such that f is not differentiable at x in some direction, is Haar null).

Previous results concerning Haar null sets

Theorem (Christensen)

Let X be a separable Banach space and $f : X \to \mathbb{R}$ a Lipschitz function. Then f is Gâteaux differentiable almost everywhere (that is, the set of those points $x \in X$ such that f is not differentiable at x in some direction, is Haar null).

Theorem (Christensen)

Suppose $\pi : G \to H$ is a universally measurable homomorphism from a Polish group G to a Polish group H, where H admits a 2-sided invariant metric compatible with its topology. Then π is continuous.

Previous results concerning Haar null sets

Theorem (Hunt)

The following set is Haar null in $C([0,1])$:

$$\{f \in C([0,1]) : \textit{there exists an } x \in [0,1] \textit{ such that } f'(x) \in \mathbb{R}\}.$$

Previous results concerning Haar null sets

Theorem (Hunt)

The following set is Haar null in $C([0,1])$*:*

$$\{f \in C([0,1]) : \textit{there exists an } x \in [0,1] \textit{ such that } f'(x) \in \mathbb{R}\}.$$

Remark

The analogous statement is true for the σ-ideal of meager sets.

Previous results concerning Haar null sets

Theorem (Hunt)

The following set is Haar null in $C([0,1])$*:*

$$\{f \in C([0,1]) : \textit{there exists an } x \in [0,1] \textit{ such that } f'(x) \in \mathbb{R}\}.$$

Remark

The analogous statement is true for the σ-ideal of meager sets.

Theorem (Dougherty-Mycielski)

The conjugacy class of $f \in S_\infty$ *is Haar positive (that is, not Haar null) if and only if* f *contains infinitely many infinite and finitely many finite cycles. Moreover, the union of all the Haar null conjugacy classes is still Haar null.*

Previous results concerning Haar null sets

Theorem (Hunt)

The following set is Haar null in $C([0,1])$:

$$\{f \in C([0,1]) : \text{there exists an } x \in [0,1] \text{ such that } f'(x) \in \mathbb{R}\}.$$

Remark

The analogous statement is true for the σ-ideal of meager sets.

Theorem (Dougherty-Mycielski)

The conjugacy class of $f \in S_\infty$ is Haar positive (that is, not Haar null) if and only if f contains infinitely many infinite and finitely many finite cycles. Moreover, the union of all the Haar null conjugacy classes is still Haar null.

Remark

There is a comeager conjugacy class in S_∞ with infinitely many finite and no infinite cycles.

Haar positive conjugacy classes in $\mathrm{Homeo}^+([0,1])$

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of $f \in \mathrm{Homeo}^+([0,1])$ is Haar positive if and only if the set of its fixed points does not have a limit point in $(0,1)$, and inside $(0,1)$, it only has “intersecting” fixed points.

Haar positive conjugacy classes in $\mathrm{Homeo}^+([0,1])$

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of $f \in \mathrm{Homeo}^+([0,1])$ *is Haar positive if and only if the set of its fixed points does not have a limit point in* $(0,1)$*, and inside* $(0,1)$*, it only has "intersecting" fixed points.*

Proof.

(Sketch of the "only if" part.) First let
$\mathcal{L} = \{f \in \mathrm{Homeo}^+([0,1]) : \mathrm{Fix}(f) \text{ has no limit points in } (0,1)\}$, we want to show that $\mathcal{L}$ is co-Haar null.

Haar positive conjugacy classes in $\text{Homeo}^+([0,1])$

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of $f \in \text{Homeo}^+([0,1])$ is Haar positive if and only if the set of its fixed points does not have a limit point in $(0,1)$, and inside $(0,1)$, it only has "intersecting" fixed points.

Proof.

(Sketch of the "only if" part.) First let $\mathcal{L} = \{f \in \text{Homeo}^+([0,1]) : \text{Fix}(f) \text{ has no limit points in } (0,1)\}$, we want to show that $\mathcal{L}$ is co-Haar null. Our probability measure to do so, is concentrated on the piecewise linear functions

$$f_a(x) = \begin{cases} 2xa & \text{if } 0 \le x < \frac{1}{2}, \\ 2(1-a)x + 2a - 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

for $a \in [1/4, 3/4]$. Thus let

$$\mu(\mathcal{B}) = 2\lambda(\Phi^{-1}(\mathcal{B})) = 2\lambda(\{a : f_a \in \mathcal{B}\}).$$

for a Borel set $B \subset \text{Homeo}^+([0,1])$.

Haar positive conjugacy classes in $\mathrm{Homeo}^+([0,1])$

Proof.

Our task is to show that $\mu(g\mathcal{L}h) = 1$ for every $g, h \in \mathrm{Homeo}^+([0,1])$. Since $\mathcal{L}$ is conjugacy invariant, $g\mathcal{L}h = gh\mathcal{L}h^{-1}h = gh\mathcal{L}$, hence it is enough to show that $\mu(g\mathcal{L}) = 1$ for every $g \in \mathrm{Homeo}^+([0,1])$, or equivalently, that $g^{-1}f_a \in \mathcal{L}$ for almost all $a \in [1/4, 3/4]$.

Proof.

Our task is to show that $\mu(g\mathcal{L}h) = 1$ for every $g, h \in \mathrm{Homeo}^+([0,1])$. Since $\mathcal{L}$ is conjugacy invariant, $g\mathcal{L}h = gh\mathcal{L}h^{-1}h = gh\mathcal{L}$, hence it is enough to show that $\mu(g\mathcal{L}) = 1$ for every $g \in \mathrm{Homeo}^+([0,1])$, or equivalently, that $g^{-1}f_a \in \mathcal{L}$ for almost all $a \in [1/4, 3/4]$.
If this is not the case then g intersects f_a infinitely many times in some interval $[\varepsilon, 1-\varepsilon]$ for positively many a and some $\varepsilon > 0$. Then we use the following result of Banach:
Lemma (Banach)
If g is of bounded variation then $\{y : g^{-1}(y)$ is infinite$\}$ is of measure zero.

Haar positive conjugacy classes in $\mathrm{Homeo}^+([0,1])$

Proof.

Our task is to show that $\mu(g\mathcal{L}h) = 1$ for every $g, h \in \mathrm{Homeo}^+([0,1])$. Since $\mathcal{L}$ is conjugacy invariant, $g\mathcal{L}h = gh\mathcal{L}h^{-1}h = gh\mathcal{L}$, hence it is enough to show that $\mu(g\mathcal{L}) = 1$ for every $g \in \mathrm{Homeo}^+([0,1])$, or equivalently, that $g^{-1}f_a \in \mathcal{L}$ for almost all $a \in [1/4, 3/4]$.
If this is not the case then g intersects f_a infinitely many times in some interval $[\varepsilon, 1-\varepsilon]$ for positively many a and some $\varepsilon > 0$. Then we use the following result of Banach:
Lemma (Banach)
If g is of bounded variation then $\{y : g^{-1}(y) \text{ is infinite}\}$ is of measure zero.
To show that the set of homeomorphisms containing only "intersecting" fixed points is also co-Haar null, use the same measure and apply ideas from the proof of the fact that a function $f : [0,1] \to \mathbb{R}$ can only have countably many strict local maximum or minimum. □

Haar positive conjugacy classes in $\text{Homeo}^+([0,1])$

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of f $\in \text{Homeo}^+([0,1])$ *is Haar positive if and only if the set of its fixed points does not have a limit point in* $(0,1)$ *and inside* $(0,1)$*, it only has "intersecting" fixed points.*

Corollary

$\text{Homeo}^+([0,1])$ *has countably infinitely many Haar positive conjugacy classes.*

Haar positive conjugacy classes in $\mathrm{Homeo}^+([0,1])$

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of $f \in \mathrm{Homeo}^+([0,1])$ is Haar positive if and only if the set of its fixed points does not have a limit point in $(0,1)$ and inside $(0,1)$, it only has "intersecting" fixed points.

Corollary

$\mathrm{Homeo}^+([0,1])$ *has countably infinitely many Haar positive conjugacy classes.*

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The union of the Haar null conjugacy classes in $\mathrm{Homeo}^+([0,1])$ is Haar null.

Haar positive conjugacy classes in $\text{Homeo}^+([0,1])$

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of $f \in \text{Homeo}^+([0,1])$ *is Haar positive if and only if the set of its fixed points does not have a limit point in* $(0,1)$ *and inside* $(0,1)$*, it only has "intersecting" fixed points.*

Corollary

$\text{Homeo}^+([0,1])$ *has countably infinitely many Haar positive conjugacy classes.*

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The union of the Haar null conjugacy classes in $\text{Homeo}^+([0,1])$ *is Haar null.*

Remark

In the Baire category sense, there is a comeager conjugacy class.

A related problem

Fact

Every (uncountable) locally compact topological group can be written as a union of a meager set and a set of Haar measure zero.

A related problem

Fact

Every (uncountable) locally compact topological group can be written as a union of a meager set and a set of Haar measure zero.

Question (Darji)

Is it true that every (uncountable) Polish group can be written as the union of a meager and a Haar null set?

A related problem

Fact

Every (uncountable) locally compact topological group can be written as a union of a meager set and a set of Haar measure zero.

Question (Darji)

Is it true that every (uncountable) Polish group can be written as the union of a meager and a Haar null set?

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

$\mathrm{Homeo}^+([0,1])$ *can be written as the union of a meager and a Haar null set.*

Haar positive conjugacy classes in $\text{Homeo}^+(S^1)$

Now we consider the group of order preserving homeomorphisms of the unit circle ($S^1 = \mathbb{R}/\mathbb{Z}$). To characterize Haar positive conjugacy classes, we need to understand conjugation.

Haar positive conjugacy classes in $\text{Homeo}^+(S^1)$

Now we consider the group of order preserving homeomorphisms of the unit circle ($S^1 = \mathbb{R}/\mathbb{Z}$). To characterize Haar positive conjugacy classes, we need to understand conjugation. A *lift* of a homeomorphism $f \in \text{Homeo}^+(S^1)$ is a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that

$$x \in \mathbb{R} \Rightarrow F(x+1) = F(x) + 1,$$
$$x \in [0,1) \Rightarrow f(x) = F(x) + k \text{ (for some } k \in \mathbb{Z}).$$

Haar positive conjugacy classes in $\mathrm{Homeo}^+(S^1)$

Now we consider the group of order preserving homeomorphisms of the unit circle ($S^1 = \mathbb{R}/\mathbb{Z}$). To characterize Haar positive conjugacy classes, we need to understand conjugation. A *lift* of a homeomorphism $f \in \mathrm{Homeo}^+(S^1)$ is a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that

$$x \in \mathbb{R} \Rightarrow F(x+1) = F(x) + 1,$$
$$x \in [0,1) \Rightarrow f(x) = F(x) + k \text{ (for some } k \in \mathbb{Z}).$$

Let

$$\tau(F) = \lim_{n\to\infty} \frac{1}{n}(F^n(x) - F(x)).$$

It is well-known that $\tau(F) - \tau(F') \in \mathbb{Z}$ for two lifts F and F' of a homeomorphism $f \in \mathrm{Homeo}^+(S^1)$. So it makes sense to define the *rotation number* of f as

$$\tau(f) = \tau(F) \pmod 1 \in \mathbb{R}/\mathbb{Z}.$$

Haar positive conjugacy classes in Homeo$^+(S^1)$

It is known that $\tau(f) \in \mathbb{Q}$ if and only if f has a periodic point, moreover, if $\tau(f) = p/q$, where $(p, q) = 1$, $q \geq 1$, then f^q has a fixed point. It is also well-known that the rotation number is conjugacy invariant.

Haar positive conjugacy classes in $\mathrm{Homeo}^+(S^1)$

It is known that $\tau(f) \in \mathbb{Q}$ if and only if f has a periodic point, moreover, if $\tau(f) = p/q$, where $(p, q) = 1$, $q \geq 1$, then f^q has a fixed point. It is also well-known that the rotation number is conjugacy invariant.

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of a homeomorphism $f \in \mathrm{Homeo}^+(S^1)$ is Haar positive if and only if $\tau(f) \in \mathbb{Q}$, it has finitely many periodic points, and if $\tau(f) = p/q$, ($(p, q) = 1$, $q \geq 1$) then f^q only has "intersecting" fixed points.

Haar positive conjugacy classes in $\text{Homeo}^+(S^1)$

It is known that $\tau(f) \in \mathbb{Q}$ if and only if f has a periodic point, moreover, if $\tau(f) = p/q$, where $(p, q) = 1$, $q \geq 1$, then f^q has a fixed point. It is also well-known that the rotation number is conjugacy invariant.

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

The conjugacy class of a homeomorphism $f \in \text{Homeo}^+(S^1)$ is Haar positive if and only if $\tau(f) \in \mathbb{Q}$, it has finitely many periodic points, and if $\tau(f) = p/q$, ($(p, q) = 1$, $q \geq 1$) then f^q only has "intersecting" fixed points.

Question

Is the union of Haar null conjugacy classes also Haar null in $\text{Homeo}^+(S^1)$?

Haar positive conjugacy classes in $\mathcal{U}(\ell^2)$

For the group of the unitary transformations of the separable Hilbert space ℓ^2 we have a partial result. The n-shift, σ_n for $n \in \{1, 2, \dots\} \cup \{\omega\}$ is the following unitary transformation: we write a basis of ℓ^2 as $\{b_i^k : i \in \mathbb{Z}, k \in n\}$, and let $\sigma_n(b_i^k) = b_{i+1}^k$.

Haar positive conjugacy classes in $\mathcal{U}(\ell^2)$

For the group of the unitary transformations of the separable Hilbert space ℓ^2 we have a partial result. The n-shift, σ_n for $n \in \{1, 2, \dots\} \cup \{\omega\}$ is the following unitary transformation: we write a basis of ℓ^2 as $\{b_i^k : i \in \mathbb{Z}, k \in n\}$, and let $\sigma_n(b_i^k) = b_{i+1}^k$.

Theorem (Darji-Elekes-Kalina-K-Vidnyánszky)

If the unitary transformation $U \in \mathcal{U}(\ell^2)$ is not conjugated to the n-shift for any n then its conjugacy class is Haar null.

Corollary

There are at most countably many Haar positive conjugacy classes in $\mathcal{U}(\ell^2)$.