Descriptive graph combinatorics

Alexander S. Kechris

Prague; July 2016

Introduction

This talk is about a relatively new subject, developed in the last two decades or so, which is at the interface of descriptive set theory and graph theory but also has interesting connections with other areas such as ergodic theory and probability theory.

Introduction

This talk is about a relatively new subject, developed in the last two decades or so, which is at the interface of descriptive set theory and graph theory but also has interesting connections with other areas such as ergodic theory and probability theory.

The object of study is the theory of definable graphs, usually Borel or analytic, on standard Borel spaces (Polish spaces with their Borel structure) and one investigates how combinatorial concepts, such as colorings and matchings, behave under definability constraints, i.e., when they are required to be definable or perhaps well-behaved in the topological or measure theoretic sense.

Introduction

Although there were a few isolated results that can now be viewed as belonging to this theory, the first systematic study of definable combinatorics appears in the paper:

Introduction

Although there were a few isolated results that can now be viewed as belonging to this theory, the first systematic study of definable combinatorics appears in the paper:

A.S. Kechris, S. Solecki and S. Todorcevic, Borel chromatic numbers, Advances in Math., 141 (1999), 1-44

Introduction

Although there were a few isolated results that can now be viewed as belonging to this theory, the first systematic study of definable combinatorics appears in the paper:

A.S. Kechris, S. Solecki and S. Todorcevic, Borel chromatic numbers, Advances in Math., 141 (1999), 1-44

A comprehensive survey of the state of the art in this area can be found in the preprint (posted in my web page):

Introduction

Although there were a few isolated results that can now be viewed as belonging to this theory, the first systematic study of definable combinatorics appears in the paper:

A.S. Kechris, S. Solecki and S. Todorcevic, Borel chromatic numbers, Advances in Math., 141 (1999), 1-44

A comprehensive survey of the state of the art in this area can be found in the preprint (posted in my web page):

A.S. Kechris and A. S. Marks, Descriptive Graph Combinatorics, preprint, 2016

Introduction

Although there were a few isolated results that can now be viewed as belonging to this theory, the first systematic study of definable combinatorics appears in the paper:

A.S. Kechris, S. Solecki and S. Todorcevic, Borel chromatic numbers, Advances in Math., 141 (1999), 1-44

A comprehensive survey of the state of the art in this area can be found in the preprint (posted in my web page):

A.S. Kechris and A. S. Marks, Descriptive Graph Combinatorics, preprint, 2016

Instead of a systematic exposition, which would take too long, I will discuss today a few representative results in this theory that give the flavor of the subject.

Chromatic numbers

A coloring of a graph $G = (V, E)$ is a map from the set of vertices V of G to a set C (the set of colors) such that adjacent vertices are assigned different colors. The chromatic number of the graph G, $\chi(G)$, is the smallest cardinality of such a C.

Chromatic numbers

A coloring of a graph $G = (V, E)$ is a map from the set of vertices V of G to a set C (the set of colors) such that adjacent vertices are assigned different colors. The chromatic number of the graph G, $\chi(G)$, is the smallest cardinality of such a C.

A graph G is bipartite if the vertices can be split into two disjoint sets $V = A \sqcup B$ such that that edges only connect vertices between A and B. This is equivalent to $\chi(G) \leq 2$. It is also equivalent to the non-existence of odd cycles. In particular, every acyclic graph is bipartite.

Borel chromatic numbers

Suppose now $G = (V, E)$ is a Borel graph (i.e., V is a standard Borel space and E is a Borel set in V^2). A Borel coloring of the graph $G = (V, E)$ is a Borel map from the set of vertices V of G to a standard Borel space C (the set of colors) such that adjacent vertices are assigned different colors. The Borel chromatic number of the graph G, $\chi_B(G)$, is the smallest cardinality of such a C. It is thus equal to one of

$$1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}.$$

Borel chromatic numbers

Suppose now $G = (V, E)$ is a Borel graph (i.e., V is a standard Borel space and E is a Borel set in V^2). A Borel coloring of the graph $G = (V, E)$ is a Borel map from the set of vertices V of G to a standard Borel space C (the set of colors) such that adjacent vertices are assigned different colors. The Borel chromatic number of the graph G, $\chi_B(G)$, is the smallest cardinality of such a C. It is thus equal to one of

$$1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}.$$

Given a probability Borel measure μ on V, we similarly define the measurable chromatic number of G, $\chi_\mu(G)$, and if V is a Polish space we define the Baire measurable chromatic number of G, $\chi_{BM}(G)$.

Borel chromatic numbers

Theorem

Borel chromatic numbers

Theorem

i) (K-Solecki-Todorcevic) There are locally countable, acyclic Borel graphs, which therefore have chromatic number 2, with Borel chromatic number $\aleph_0, 2^{\aleph_0}$.

Borel chromatic numbers

Theorem

i) (K-Solecki-Todorcevic) There are locally countable, acyclic Borel graphs, which therefore have chromatic number 2, with Borel chromatic number $\aleph_0, 2^{\aleph_0}$.
ii) (KST) Every locally finite Borel graph has Borel chromatic number $\leq \aleph_0$. *There are l.f., acyclic Borel graphs with Borel chromatic number* $\aleph_0$.

Borel chromatic numbers

Theorem

i) (K-Solecki-Todorcevic) There are locally countable, acyclic Borel graphs, which therefore have chromatic number 2, with Borel chromatic number $\aleph_0, 2^{\aleph_0}$.
ii) (KST) Every locally finite Borel graph has Borel chromatic number $\leq \aleph_0$. *There are l.f., acyclic Borel graphs with Borel chromatic number* $\aleph_0$.
iii) (KST) Every Borel graph with bounded degree $\leq d$ *has Borel chromatic number* $\leq d+1$.

Borel chromatic numbers

Theorem

i) (K-Solecki-Todorcevic) There are locally countable, acyclic Borel graphs, which therefore have chromatic number 2, with Borel chromatic number $\aleph_0, 2^{\aleph_0}$.
ii) (KST) Every locally finite Borel graph has Borel chromatic number $\leq \aleph_0$. *There are l.f., acyclic Borel graphs with Borel chromatic number* $\aleph_0$.
iii) (KST) Every Borel graph with bounded degree $\leq d$ *has Borel chromatic number* $\leq d+1$.*(Conley-K, 2009) There are bounded degree, acyclic Borel graphs whose Borel chromatic number takes any finite value.*

Borel chromatic numbers

Theorem

i) (K-Solecki-Todorcevic) There are locally countable, acyclic Borel graphs, which therefore have chromatic number 2, with Borel chromatic number $\aleph_0, 2^{\aleph_0}$.
ii) (KST) Every locally finite Borel graph has Borel chromatic number $\leq \aleph_0$. *There are l.f., acyclic Borel graphs with Borel chromatic number* $\aleph_0$.
iii) (KST) Every Borel graph with bounded degree $\leq d$ *has Borel chromatic number* $\leq d+1$.*(Conley-K, 2009) There are bounded degree, acyclic Borel graphs whose Borel chromatic number takes any finite value. (Marks, 2015) There are* d*-regular, acyclic Borel graphs whose Borel chromatic number takes any value in* $\{1, 2, \ldots, d+1\}$.

Of special interest are graphs generated by group actions. Let (Γ, S) be a marked group, i.e, a group with a finite, symmetric set of generators S. If a is a free Borel action of Γ on a standard Borel space V this gives rise to a Borel graph on V, the "Cayley graph" of the action, where two vertices $x, y \in V$ are connected iff a generator $s \in S$ sends x to y.

Borel chromatic numbers of shift graphs

Of special interest are graphs generated by group actions. Let (Γ, S) be a marked group, i.e, a group with a finite, symmetric set of generators S. If a is a free Borel action of Γ on a standard Borel space V this gives rise to a Borel graph on V, the "Cayley graph" of the action, where two vertices $x, y \in V$ are connected iff a generator $s \in S$ sends x to y.

Every connected component of this graph is a copy of the Cayley graph of (Γ, S), so this graph has the same chromatic number as the Cayley graph of the group. However the Borel chromatic number behaves very differently and reflects the complexity of the group and the action.

Borel chromatic numbers of shift graphs

Consider the shift action of the group Γ on $[0,1]^\Gamma$, restricted to its free part. Denote its "Cayley graph" by $G_\infty(\Gamma, S)$. On general grounds this has the highest Borel chromatic number among all free actions of Γ and it is bounded by $d+1$, where $d = |S|$.

Borel chromatic numbers of shift graphs

Consider the shift action of the group Γ on $[0,1]^\Gamma$, restricted to its free part. Denote its "Cayley graph" by $G_\infty(\Gamma, S)$. On general grounds this has the highest Borel chromatic number among all free actions of Γ and it is bounded by $d+1$, where $d = |S|$.

Take now the groups $\mathbb{Z}^n$ and $\mathbb{F}_n$, with their usual set of generators S, which we will not explicitly indicate below. The graphs $G_\infty(\mathbb{Z}^n), G_\infty(\mathbb{F}_n)$ are both bipartite, so they have chromatic number 2. But we have two contrasting pictures when we look at the Borel chromatic numbers:

Borel chromatic numbers of shift graphs

Consider the shift action of the group Γ on $[0,1]^\Gamma$, restricted to its free part. Denote its "Cayley graph" by $G_\infty(\Gamma, S)$. On general grounds this has the highest Borel chromatic number among all free actions of Γ and it is bounded by $d+1$, where $d = |S|$.

Take now the groups $\mathbb{Z}^n$ and $\mathbb{F}_n$, with their usual set of generators S, which we will not explicitly indicate below. The graphs $G_\infty(\mathbb{Z}^n), G_\infty(\mathbb{F}_n)$ are both bipartite, so they have chromatic number 2. But we have two contrasting pictures when we look at the Borel chromatic numbers:

Theorem (Conley-K, Lyons-Nazarov, 2009)

$\chi_B(G_\infty(\mathbb{F}_n)) \to \infty$, *as* $n \to \infty$

Borel chromatic numbers of shift graphs

Consider the shift action of the group Γ on $[0,1]^\Gamma$, restricted to its free part. Denote its "Cayley graph" by $G_\infty(\Gamma, S)$. On general grounds this has the highest Borel chromatic number among all free actions of Γ and it is bounded by $d+1$, where $d = |S|$.

Take now the groups $\mathbb{Z}^n$ and $\mathbb{F}_n$, with their usual set of generators S, which we will not explicitly indicate below. The graphs $G_\infty(\mathbb{Z}^n), G_\infty(\mathbb{F}_n)$ are both bipartite, so they have chromatic number 2. But we have two contrasting pictures when we look at the Borel chromatic numbers:

Theorem (Conley-K, Lyons-Nazarov, 2009)

$\chi_B(G_\infty(\mathbb{F}_n)) \to \infty$, *as* $n \to \infty$

Theorem (Gao-Jackson-Krohne-Seward, 2015)

$\chi_B(G_\infty(\mathbb{Z}^n)) = 3$

Borel chromatic numbers of shift graphs

More recently Marks computed exactly $\chi_B(G_\infty(\mathbb{F}_n))$.

Borel chromatic numbers of shift graphs

More recently Marks computed exactly $\chi_B(G_\infty(\mathbb{F}_n))$.

Theorem (Marks, 2015)

$\chi_B(G_\infty(\mathbb{F}_n)) = 2n + 1$

Borel chromatic numbers of shift graphs

More recently Marks computed exactly $\chi_B(G_\infty(\mathbb{F}_n))$.

Theorem (Marks, 2015)

$\chi_B(G_\infty(\mathbb{F}_n)) = 2n + 1$

What about the measurable chromatic number $\chi_\mu(G_\infty(\mathbb{F}_n))$ of the shift graph, where μ is the usual product measure?

Borel chromatic numbers of shift graphs

More recently Marks computed exactly $\chi_B(G_\infty(\mathbb{F}_n))$.

Theorem (Marks, 2015)

$\chi_B(G_\infty(\mathbb{F}_n)) = 2n + 1$

What about the measurable chromatic number $\chi_\mu(G_\infty(\mathbb{F}_n))$ of the shift graph, where μ is the usual product measure?

Theorem (K-Marks, 2015)

$\chi_\mu(G_\infty(\mathbb{F}_n)) \geq \max(3, \frac{n}{\log 2n})$

Borel chromatic numbers of shift graphs

More recently Marks computed exactly $\chi_B(G_\infty(\mathbb{F}_n))$.

Theorem (Marks, 2015)

$\chi_B(G_\infty(\mathbb{F}_n)) = 2n + 1$

What about the measurable chromatic number $\chi_\mu(G_\infty(\mathbb{F}_n))$ of the shift graph, where μ is the usual product measure?

Theorem (K-Marks, 2015)

$\chi_\mu(G_\infty(\mathbb{F}_n)) \geq \max(3, \frac{n}{\log 2n})$

Very recently the following upper bound was proved

Borel chromatic numbers of shift graphs

More recently Marks computed exactly $\chi_B(G_\infty(\mathbb{F}_n))$.

Theorem (Marks, 2015)

$\chi_B(G_\infty(\mathbb{F}_n)) = 2n + 1$

What about the measurable chromatic number $\chi_\mu(G_\infty(\mathbb{F}_n))$ of the shift graph, where μ is the usual product measure?

Theorem (K-Marks, 2015)

$\chi_\mu(G_\infty(\mathbb{F}_n)) \geq \max(3, \frac{n}{\log 2n})$

Very recently the following upper bound was proved

Theorem (Bernshteyn, 2016)

$\chi_\mu(G_\infty(\mathbb{F}_n)) \leq (1 + o(1))\frac{2n}{\log 2n}$

Borel chromatic numbers of shift graphs

Thus

$$\frac{n}{\log 2n} \leq \chi_\mu(G_\infty(\mathbb{F}_n)) \leq (1 + o(1))\frac{2n}{\log 2n}$$

Thus

$$\frac{n}{\log 2n} \leq \chi_\mu(G_\infty(\mathbb{F}_n)) \leq (1 + o(1))\frac{2n}{\log 2n}$$

and

$$\frac{\chi_B(G_\infty(\mathbb{F}_n))}{\chi_\mu(G_\infty(\mathbb{F}_n))} \asymp \log n$$

Thus

$$\frac{n}{\log 2n} \leq \chi_\mu(G_\infty(\mathbb{F}_n)) \leq (1+o(1))\frac{2n}{\log 2n}$$

and

$$\frac{\chi_B(G_\infty(\mathbb{F}_n))}{\chi_\mu(G_\infty(\mathbb{F}_n))} \asymp \log n$$

Problem

Calculate $\chi_\mu(G_\infty(\mathbb{F}_n))$

Borel chromatic numbers of shift graphs

Thus

$$\frac{n}{\log 2n} \leq \chi_\mu(G_\infty(\mathbb{F}_n)) \leq (1+o(1))\frac{2n}{\log 2n}$$

and

$$\frac{\chi_B(G_\infty(\mathbb{F}_n))}{\chi_\mu(G_\infty(\mathbb{F}_n))} \asymp \log n$$

Problem

Calculate $\chi_\mu(G_\infty(\mathbb{F}_n))$

By contrast, Conley and B. Miller (2014) have shown that the Baire measurable chromatic number $\chi_{BM}(G_\infty(\mathbb{F}_n))$ is also equal to 3.

$1, 2, \ldots, 2n+1, \aleph_0$

$1, 2, \ldots, 2n+1, \aleph_0$

Let $f_1, f_2, \ldots, f_n$ be Borel functions on a standard Borel space V. Consider the Borel graph $G_{f_1, f_2, \ldots, f_n}$ with vertex set V, where $x, y \in V$ are connected by an edge iff there is $i \leq n$ such that $f_i(x) = y$ or $f_i(y) = x$. (Equivalently this is the undirected version of a directed Borel graph of out-degree $\leq n$.) What is the Borel chromatic number of this graph?

$1, 2, \ldots, 2n+1, \aleph_0$

Let $f_1, f_2, \ldots, f_n$ be Borel functions on a standard Borel space V. Consider the Borel graph $G_{f_1, f_2, \ldots, f_n}$ with vertex set V, where $x, y \in V$ are connected by an edge iff there is $i \leq n$ such that $f_i(x) = y$ or $f_i(y) = x$. (Equivalently this is the undirected version of a directed Borel graph of out-degree $\leq n$.) What is the Borel chromatic number of this graph?

Problem (K-Solecki-Todorcevic)

$\chi_B(G_{f_1, f_2, \ldots, f_n})$ *is one of* $1, 2, \ldots, 2n+1, \aleph_0$.

$1, 2, \ldots, 2n+1, \aleph_0$

Let $f_1, f_2, \ldots, f_n$ be Borel functions on a standard Borel space V. Consider the Borel graph $G_{f_1, f_2, \ldots, f_n}$ with vertex set V, where $x, y \in V$ are connected by an edge iff there is $i \leq n$ such that $f_i(x) = y$ or $f_i(y) = x$. (Equivalently this is the undirected version of a directed Borel graph of out-degree $\leq n$.) What is the Borel chromatic number of this graph?

Problem (K-Solecki-Todorcevic)

$\chi_B(G_{f_1, f_2, \ldots, f_n})$ *is one of* $1, 2, \ldots, 2n+1, \aleph_0$.

We have $\chi_B(G_{f_1, f_2, \ldots, f_n}) \leq \aleph_0$ (KST). For finite V the possible values (of the chromatic number) are exactly $1, 2, \ldots, 2n+1$ but for Borel graphs we have one more possibility:

$1, 2, \ldots, 2n+1, \aleph_0$

Let $f_1, f_2, \ldots, f_n$ be Borel functions on a standard Borel space V. Consider the Borel graph $G_{f_1,f_2,\ldots,f_n}$ with vertex set V, where $x, y \in V$ are connected by an edge iff there is $i \leq n$ such that $f_i(x) = y$ or $f_i(y) = x$. (Equivalently this is the undirected version of a directed Borel graph of out-degree $\leq n$.) What is the Borel chromatic number of this graph?

Problem (K-Solecki-Todorcevic)

$\chi_B(G_{f_1,f_2,\ldots,f_n})$ *is one of* $1, 2, \ldots, 2n+1, \aleph_0$.

We have $\chi_B(G_{f_1,f_2,\ldots,f_n}) \leq \aleph_0$ (KST). For finite V the possible values (of the chromatic number) are exactly $1, 2, \ldots, 2n+1$ but for Borel graphs we have one more possibility:

Example (KST)

Consider the space V of all increasing sequences of natural numbers and let s be the shift map. Then $\chi_B(G_s) = \aleph_0$.

$1, 2, \ldots, 2n+1, \aleph_0$

Here is what is known about this problem:

$1, 2, \ldots, 2n+1, \aleph_0$

Here is what is known about this problem:

Theorem

- *The answer is positive for* $n = 1$ *(K-Solecki-Todorcevic); also for* $n = 2$ *and almost for* $n = 3$ *(with 8 instead of the optimal 7) (Palamourdas, 2012).*

$1, 2, \ldots, 2n+1, \aleph_0$

Here is what is known about this problem:

Theorem

- *The answer is positive for* $n = 1$ *(K-Solecki-Todorcevic); also for* $n = 2$ *and almost for* $n = 3$ *(with 8 instead of the optimal 7) (Palamourdas, 2012).*
- *The answer is positive for every* n *if the functions are commuting and fixed-point free (Palamourdas, 2012).*

$1, 2, \ldots, 2n+1, \aleph_0$

Here is what is known about this problem:

Theorem

- *The answer is positive for* $n = 1$ *(K-Solecki-Todorcevic); also for* $n = 2$ *and almost for* $n = 3$ *(with 8 instead of the optimal 7) (Palamourdas, 2012).*
- *The answer is positive for every* n *if the functions are commuting and fixed-point free (Palamourdas, 2012).*
- *For each* $n \geq 3$*, the Borel chromatic number is in the set:* $\{1, 2, \ldots, \frac{1}{2}(n+1)(n+2) - 2, \aleph_0\}$ *(Palamourdas, 2012; Meehan, 2015).*

$1, 2, \ldots, 2n+1, \aleph_0$

Here is what is known about this problem:

Theorem

- *The answer is positive for* $n = 1$ *(K-Solecki-Todorcevic); also for* $n = 2$ *and almost for* $n = 3$ *(with 8 instead of the optimal 7) (Palamourdas, 2012).*
- *The answer is positive for every* n *if the functions are commuting and fixed-point free (Palamourdas, 2012).*
- *For each* $n \geq 3$*, the Borel chromatic number is in the set:* $\{1, 2, \ldots, \frac{1}{2}(n+1)(n+2) - 2, \aleph_0\}$ *(Palamourdas, 2012; Meehan, 2015).*

Finally, B. Miller and Palamourdas showed that if one is willing to throw away a meager set or a null set (for any Borel measure), then the Borel chromatic numbers of these graphs are finite.

Vizing's Theorem

Given a graph G, its edge chromatic number, in symbols, $\chi'(G)$, is the smallest number of colors that we can use to color the edges of the graph so that adjacent edges have different colors. For a Borel graph, we define similarly its Borel edge chromatic number, $\chi'_B(G)$ (and $\chi'_\mu(G)$).

Vizing's Theorem

The following is a classical theorem of Vizing, which gives the optimal edge chromatic number:

Vizing's Theorem

The following is a classical theorem of Vizing, which gives the optimal edge chromatic number:

Theorem (Vizing)

For any graph of degree $\leq d$, *we have* $\chi'(G) \leq d + 1$.

Vizing's Theorem

The following is a classical theorem of Vizing, which gives the optimal edge chromatic number:

Theorem (Vizing)

For any graph of degree $\leq d$, we have $\chi'(G) \leq d + 1$.

What is the optimal Borel edge chromatic number? Is it again $d + 1$?

Vizing's Theorem

The following is a classical theorem of Vizing, which gives the optimal edge chromatic number:

Theorem (Vizing)

For any graph of degree $\leq d$, *we have* $\chi'(G) \leq d + 1$.

What is the optimal Borel edge chromatic number? Is it again $d + 1$?

Theorem

Let G *be a Borel graph of degree* $\leq d$. *Then:*

- *(K-Solecki-Todorcevic)* $\chi'_B(G) \leq 2d - 1$.

Vizing's Theorem

The following is a classical theorem of Vizing, which gives the optimal edge chromatic number:

Theorem (Vizing)

For any graph of degree $\leq d$*, we have* $\chi'(G) \leq d + 1$*.*

What is the optimal Borel edge chromatic number? Is it again $d + 1$?

Theorem

Let G *be a Borel graph of degree* $\leq d$*. Then:*

- *(K-Solecki-Todorcevic)* $\chi'_B(G) \leq 2d - 1$*.*
- *(Marks, 2015) This is optimal: There are* d*-regular acyclic Borel graphs where* $\chi'_B(G)$ *can take any value between* d *and* $2d - 1$*.*

Vizing's Theorem

The following is a classical theorem of Vizing, which gives the optimal edge chromatic number:

Theorem (Vizing)

For any graph of degree $\leq d$, we have $\chi'(G) \leq d+1$.

What is the optimal Borel edge chromatic number? Is it again $d+1$?

Theorem

Let G be a Borel graph of degree $\leq d$. Then:

- *(K-Solecki-Todorcevic) $\chi'_B(G) \leq 2d-1$.*
- *(Marks, 2015) This is optimal: There are d-regular acyclic Borel graphs where $\chi'_B(G)$ can take any value between d and $2d-1$.*

Thus, surprisingly, the optimal value in the Borel problem is $2d-1$ instead of $d+1$ colors.

Vizing's Theorem

On the other hand for measurable edge chromatic numbers we have

Vizing's Theorem

On the other hand for measurable edge chromatic numbers we have

Theorem (Csóka-Lippner-Pikhurko, 2014)

Let G be a Borel graph of degree $\leq d$ and let μ be such that G is measure-preserving. Then

- $\chi'_\mu(G) \leq d + O(\sqrt{d})$
- *If G is bipartite, then $\chi'_\mu(G) \leq d + 1$*

König's Theorem

A matching in a graph $G = (V, E)$ is a set M of edges that have no common vertex. For a matching M denote by V_M the set of matched vertices and call M a perfect matching if $V_M = V$. If a measure μ on V is present and V_M has full measure, we say that M is a perfect matching μ-a.e.

König's Theorem

A matching in a graph $G = (V, E)$ is a set M of edges that have no common vertex. For a matching M denote by V_M the set of matched vertices and call M a perfect matching if $V_M = V$. If a measure μ on V is present and V_M has full measure, we say that M is a perfect matching μ-a.e.

Theorem (König)

Every d-regular bipartite graph has a perfect matching, for any $d \geq 2$.

A. Miller's Problem

In the 1980's Arnie Miller asked whether König's Theorem holds in the Borel category:

A. Miller's Problem

In the 1980's Arnie Miller asked whether König's Theorem holds in the Borel category:

Problem (A. Miller)

Let $G = (V, E)$ be a Borel d-regular, Borel bipartite graph. Is it true that G has a Borel perfect matching?

The case $d = 2$

Laczkovich (1988) obtained a negative answer for $d = 2$.

The case $d = 2$

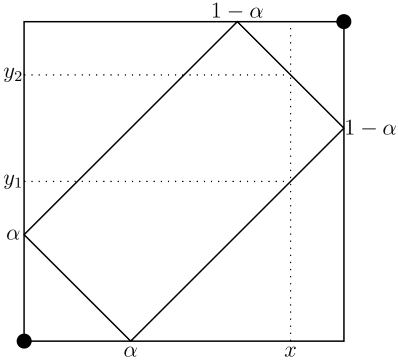
Laczkovich (1988) obtained a negative answer for $d = 2$.

Here is his example:

Laczkovich (1988) obtained a negative answer for $d = 2$.

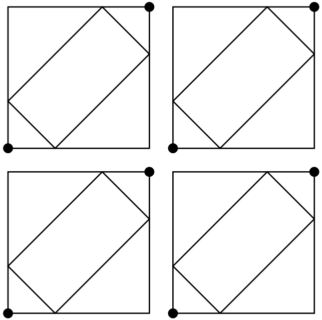
Here is his example:

Fix an irrational $0 < \alpha < 1$ and consider the set consisting of the following rectangle inscribed in the unit square, together with the indicated two corner points.



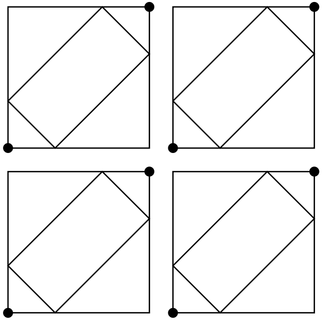
The even d case

In a paper in the early 2000s it was "shown" that putting together 4 copies of the preceding graph would produce examples for $d = 4$ (and similarly for any even d).



The even d case

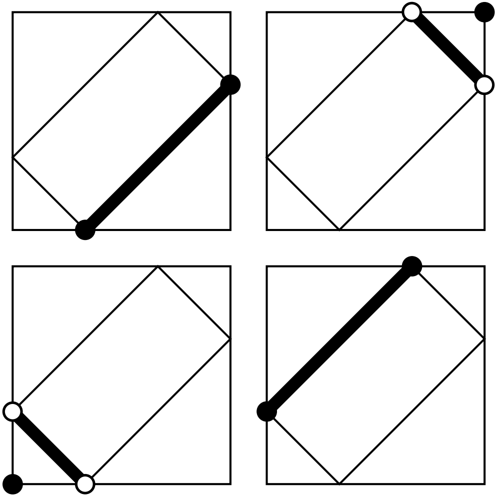
In a paper in the early 2000s it was "shown" that putting together 4 copies of the preceding graph would produce examples for $d = 4$ (and similarly for any even d).



But around 2009 Lyons showed that this did not work as this graph had a Borel matching.

The even d case

Here is a simple perfect matching found later by Conley-K:

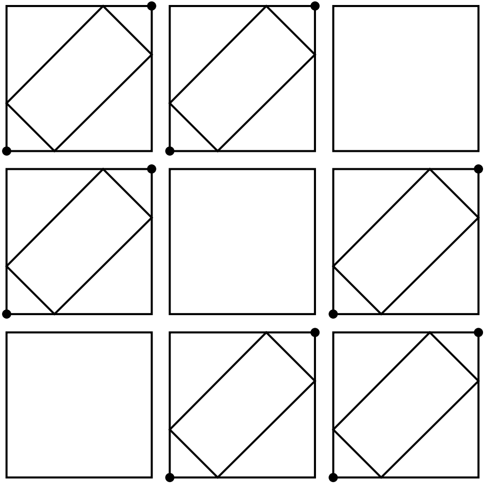


The even d case

Luckily Conley-K found a way to salvage this approach by using a "Sudoku" version:

The even d case

Luckily Conley-K found a way to salvage this approach by using a "Sudoku" version:



The general d case

These ideas do not work for odd d, so Conley-K (2009) suggested a different approach based also on ergodic theory. Let $\mathbb{Z}_d$ be the cyclic group of order d, let $A = B = \mathbb{Z}_d$ and consider the free part of the shift action of $A * B$ on $[0, 1]^{A*B}$. This gives a d-regular, Borel acyclic, Borel bipartite graph G_d with the one side of the graph consisting of the A-orbits and the other side consisting of the B-orbits. Two such orbits are connected by an edge iff they intersect.

The general d case

These ideas do not work for odd d, so Conley-K (2009) suggested a different approach based also on ergodic theory. Let $\mathbb{Z}_d$ be the cyclic group of order d, let $A = B = \mathbb{Z}_d$ and consider the free part of the shift action of $A * B$ on $[0,1]^{A*B}$. This gives a d-regular, Borel acyclic, Borel bipartite graph G_d with the one side of the graph consisting of the A-orbits and the other side consisting of the B-orbits. Two such orbits are connected by an edge iff they intersect.

For $d = 2$ this has no Borel matching *even a.e.*, using the fact that the shift is weakly mixing.

The general d case

These ideas do not work for odd d, so Conley-K (2009) suggested a different approach based also on ergodic theory. Let $\mathbb{Z}_d$ be the cyclic group of order d, let $A = B = \mathbb{Z}_d$ and consider the free part of the shift action of $A * B$ on $[0,1]^{A*B}$. This gives a d-regular, Borel acyclic, Borel bipartite graph G_d with the one side of the graph consisting of the A-orbits and the other side consisting of the B-orbits. Two such orbits are connected by an edge iff they intersect.

For $d = 2$ this has no Borel matching *even a.e.*, using the fact that the shift is weakly mixing.

It was hoped that these ergodic theory arguments would carry over to every d but this hope was dashed by a later result of Lyons-Nazarov that showed that for $d = 3$ there is indeed a Borel perfect matching a.e.

The general d case

This was a consequence of the following result of Lyons-Nazarov.

The general d case

This was a consequence of the following result of Lyons-Nazarov.

Theorem (Lyons-Nazarov, 2009)

Let (Γ, S) be a non-amenable marked group with bipartite Cayley graph. Then $G_\infty(\Gamma, S)$ admits a Borel perfect matching a.e. (with respect to the usual product measure).

The general d case

This was a consequence of the following result of Lyons-Nazarov.

Theorem (Lyons-Nazarov, 2009)

Let (Γ, S) be a non-amenable marked group with bipartite Cayley graph. Then $G_\infty(\Gamma, S)$ admits a Borel perfect matching a.e. (with respect to the usual product measure).

We mention also here the following important improvement by Csóka-Lippner.

The general d case

This was a consequence of the following result of Lyons-Nazarov.

Theorem (Lyons-Nazarov, 2009)

Let (Γ, S) be a non-amenable marked group with bipartite Cayley graph. Then $G_\infty(\Gamma, S)$ admits a Borel perfect matching a.e. (with respect to the usual product measure).

We mention also here the following important improvement by Csóka-Lippner.

Theorem (Csóka-Lippner, 2012)

Let (Γ, S) be a non-amenable marked group. Then $G_\infty(\Gamma, S)$ admits a Borel perfect matching a.e.

The general d case

So ergodic theory cannot work to show that the graphs G_d admit no perfect matching for all d. However Marks recently used completely different methods, employing infinite games and Martin's Borel Determinacy Theorem, to finally show the following:

The general d case

So ergodic theory cannot work to show that the graphs G_d admit no perfect matching for all d. However Marks recently used completely different methods, employing infinite games and Martin's Borel Determinacy Theorem, to finally show the following:

Theorem (Marks, 2015)

The graph G_d has no Borel perfect matching, for any $d \geq 2$.

The general d case

So ergodic theory cannot work to show that the graphs G_d admit no perfect matching for all d. However Marks recently used completely different methods, employing infinite games and Martin's Borel Determinacy Theorem, to finally show the following:

Theorem (Marks, 2015)

The graph G_d has no Borel perfect matching, for any $d \geq 2$.

Remark

Borel determinacy needs quite a bit of set theoretic power as it uses (necessarily) the existence of sets of size at least the $\aleph_1$ iteration of the power set operation. Therefore, strangely, the only known proof of the preceding theorem needs to make use of these very large sets. The same comment applies to Marks' calculation of the Borel chromatic number of $G_\infty(\mathbb{F}_n)$.

Applications to paradoxical decompositions

There is a close connection between matchings and paradoxical decompositions. Thus some of the results on matchings in descriptive graph combinatorics have applications in the theory of paradoxical decompositions. I will discuss below some very recent work in this area.

Applications to paradoxical decompositions

First some basic definitions.

Applications to paradoxical decompositions

First some basic definitions.

Definition

Suppose a group G acts on a space X. If $A, B \subseteq X$, then A, B are G-equidecomposable if there are partitions $A = \bigsqcup_{i=1}^{n} A_i, B = \bigsqcup_{i=1}^{n} B_i$ into finitely many pieces and group elements $g_1, \dots, g_n$ such that $g_1 \cdot A_1 = B_1, \dots, g_n \cdot A_n = B_n$.

Applications to paradoxical decompositions

First some basic definitions.

Definition

Suppose a group G acts on a space X. If $A, B \subseteq X$, then A, B are G-equidecomposable if there are partitions $A = \bigsqcup_{i=1}^{n} A_i, B = \bigsqcup_{i=1}^{n} B_i$ into finitely many pieces and group elements $g_1, \dots, g_n$ such that $g_1 \cdot A_1 = B_1, \dots, g_n \cdot A_n = B_n$.

Definition

A subset X is G-paradoxical if there is a partition $X = A \sqcup B$ into two pieces A, B which are equidecomposable with X. Such a partition is called a paradoxical decomposition of X

Applications to paradoxical decompositions

We now have the following famous Banach-Tarski Paradox.

Applications to paradoxical decompositions

We now have the following famous Banach-Tarski Paradox.

Theorem (Banach-Tarski)

For any $n \geq 3$ (and with respect to the group of rigid motions (isometries) of $\mathbb{R}^n$), any closed ball in $\mathbb{R}^n$ is paradoxical and any two bounded subsets of $\mathbb{R}^n$ with nonempty interior are equidecomposable.

Applications to paradoxical decompositions

In the early 1990's Dougherty and Foreman solved Marczewski's Problem (from the 1930's) by showing that the Banach-Tarski Paradox can be performed using pieces with the Property of Baire. Their proof was based on the following result:

Applications to paradoxical decompositions

In the early 1990's Dougherty and Foreman solved Marczewski's Problem (from the 1930's) by showing that the Banach-Tarski Paradox can be performed using pieces with the Property of Baire. Their proof was based on the following result:

Theorem (Dougherty-Foreman, 1994)

Let the free group $\mathbb{F}_n, n \geq 2$, act freely by homeomorphisms on a Polish space X. Then X is paradoxical with pieces having the property of Baire.

Applications to paradoxical decompositions

In the early 1990's Dougherty and Foreman solved Marczewski's Problem (from the 1930's) by showing that the Banach-Tarski Paradox can be performed using pieces with the Property of Baire. Their proof was based on the following result:

Theorem (Dougherty-Foreman, 1994)

Let the free group $\mathbb{F}_n, n \geq 2$, act freely by homeomorphisms on a Polish space X. Then X is paradoxical with pieces having the property of Baire.

Another proof of this result has been recently found by K-Marks using ideas concerning matchings in descriptive graph combinatorics. Further work of Marks-Unger led to an ultimate form of the Dougherty-Foreman result.

Applications to paradoxical decompositions

The classical Hall Theorem about matchings states the following:

Theorem (Hall)

Let G be a locally finite bipartite graph such that for any finite set of vertices F (contained in one piece of the bipartition), we have $|N_G(F)| \geq |F|$. Then G admits a perfect matching.

Applications to paradoxical decompositions

The classical Hall Theorem about matchings states the following:

Theorem (Hall)

Let G be a locally finite bipartite graph such that for any finite set of vertices F (contained in one piece of the bipartition), we have $|N_G(F)| \geq |F|$. Then G admits a perfect matching.

However, the Hall condition is not enough to guarantee matchings in a measurable or generic context:

Applications to paradoxical decompositions

The classical Hall Theorem about matchings states the following:

Theorem (Hall)

Let G be a locally finite bipartite graph such that for any finite set of vertices F (contained in one piece of the bipartition), we have $|N_G(F)| \geq |F|$. Then G admits a perfect matching.

However, the Hall condition is not enough to guarantee matchings in a measurable or generic context:

Proposition

- *(K-Marks, 2015) For each $n \geq 1$, there is a bounded degree Borel bipartite graph G on a standard probability space (X, μ) that satisfies $|N_G(F)| \geq n|F|$, for any finite set of vertices F, but G has no Borel perfect matching μ-a.e.*

Applications to paradoxical decompositions

The classical Hall Theorem about matchings states the following:

Theorem (Hall)

Let G be a locally finite bipartite graph such that for any finite set of vertices F (contained in one piece of the bipartition), we have $|N_G(F)| \geq |F|$. Then G admits a perfect matching.

However, the Hall condition is not enough to guarantee matchings in a measurable or generic context:

Proposition

- *(K-Marks, 2015) For each $n \geq 1$, there is a bounded degree Borel bipartite graph G on a standard probability space (X, μ) that satisfies $|N_G(F)| \geq n|F|$, for any finite set of vertices F, but G has no Borel perfect matching μ-a.e.*
- *There is a bounded degree Borel bipartite graph G on a Polish space X that satisfies $|N_G(F)| \geq |F|$, for any finite set of vertices F, but G has no Borel perfect matching on a comeager set.*

Applications to paradoxical decompositions

However very recently Marks and Unger showed that an ϵ strengthening suffices.

Applications to paradoxical decompositions

However very recently Marks and Unger showed that an ϵ strengthening suffices.

Theorem (Marks-Unger, 2016)

Let G be a locally finite bipartite Borel graph such that for some $\epsilon > 0$ and any finite set of vertices F (contained in one piece of the bipartition), we have $|N_G(F)| \geq (1 + \epsilon)|F|$. Then G admits a perfect matching on a comeager set.

Mark and Unger then used this result to prove an ultimate form of the Dougherty-Foreman Theorem (by very different methods) and the solution of the Marczewski Problem:

Applications to paradoxical decompositions

Mark and Unger then used this result to prove an ultimate form of the Dougherty-Foreman Theorem (by very different methods) and the solution of the Marczewski Problem:

Theorem (Marks-Unger, 2016)

Suppose a group Γ acts by Borel automorphisms on a Polish space. If the action has a paradoxical decomposition, then it has one using sets with the property of Baire.

Applications to paradoxical decompositions

I will finish with some recent results on measurable versions of the Banach-Tarski Paradox.

Applications to paradoxical decompositions

I will finish with some recent results on measurable versions of the Banach-Tarski Paradox.

Using a version of the Lyons-Nazarov Theorem mentioned earlier, Grabowski-Máthé-Pikhurko have shown the following:

Applications to paradoxical decompositions

I will finish with some recent results on measurable versions of the Banach-Tarski Paradox.

Using a version of the Lyons-Nazarov Theorem mentioned earlier, Grabowski-Máthé-Pikhurko have shown the following:

Theorem (Grabowski-Máthé-Pikhurko, 2016)

Let A, B be Lebesgue measurable subsets of $\mathbb{R}^n$ with $n \geq 3$. Suppose they are bounded and have nonempty interior. They are equidecomposable by rigid motions using Lebesgue measurable pieces iff they have the same measure.

Applications to paradoxical decompositions

I will finish with some recent results on measurable versions of the Banach-Tarski Paradox.

Using a version of the Lyons-Nazarov Theorem mentioned earlier, Grabowski-Máthé-Pikhurko have shown the following:

Theorem (Grabowski-Máthé-Pikhurko, 2016)

Let A, B be Lebesgue measurable subsets of $\mathbb{R}^n$ with $n \geq 3$. Suppose they are bounded and have nonempty interior. They are equidecomposable by rigid motions using Lebesgue measurable pieces iff they have the same measure.

Moreover, using also ideas related to Laczkovich's solution of the Tarski Circle Squaring Problem, they showed the following:

Applications to paradoxical decompositions

I will finish with some recent results on measurable versions of the Banach-Tarski Paradox.
Using a version of the Lyons-Nazarov Theorem mentioned earlier, Grabowski-Máthé-Pikhurko have shown the following:

Theorem (Grabowski-Máthé-Pikhurko, 2016)

Let A, B be Lebesgue measurable subsets of $\mathbb{R}^n$ with $n \geq 3$. Suppose they are bounded and have nonempty interior. They are equidecomposable by rigid motions using Lebesgue measurable pieces iff they have the same measure.

Moreover, using also ideas related to Laczkovich's solution of the Tarski Circle Squaring Problem, they showed the following:

Theorem (Grabowski-Máthé-Pikhurko, 2015)

Let A, B be Lebesgue measurable subsets of $\mathbb{R}^n$ with $n \geq 1$. Suppose they are bounded with nonempty interior and have the same Lebesgue measure and their boundaries have box dimension less than n. Then they are equidecomposable by translations using Lebesgue measurable pieces.