Dimension of inverse limits with set-valued functions

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Abstract

Abstract : In this talk, we investigate dimension of inverse limits with set-valued functions.

Dimension of inverse limits with set-valued functions

Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function for each $i \in \mathbb{N}$. The *inverse limit* of the inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ is the space

$$\varprojlim\{X_i, f_{i,i+1}\} = \{(x_i)_{i=1}^{\infty} \mid x_i \in f_{i,i+1}(x_{i+1}) \text{ for each } i \in \mathbb{N}\} \subset \prod_{i=1}^{\infty} X_i$$

which has the topology inherited as a subspace of the product space $\prod_{i=1}^{\infty} X_i$.

In particular, if $f : X \to 2^X$ is an upper semi-continuous function, we consider the inverse sequence $\{X, f\} = \{X_i, f_{i,i+1}\}$, where $X_i = X, f_{i,i+1} = f \ (i \in \mathbb{N})$. We put

$$\varprojlim\{X, f\} = \{(x_i)_{i=1}^{\infty} \mid x_i \in f(x_{i+1}) \text{ for each } i \in \mathbb{N}\}.$$

Theorem 2.1

Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to X_i$ be a map (single valued upper semi-continuous function) for each $i \in \mathbb{N}$. Then $\dim \varprojlim \{X_i, f_{i,i+1}\} \leq \sup\{\dim X_i \mid i \in \mathbb{N}\}$.

Concerning dimension of inverse limits with set-valued functions, the following theorems have been obtained.

Theorem 2.2 (Banič)

Suppose that X is a continuum and A a closed subset of X. Let $g : X \to X$ be a (continuous) map. If $f : X \to 2^X$ is the upper semi-continuous function such that $G(f) = G(g) \cup (A \times X)$, then $\dim \varprojlim \{X, f\} \in \{\dim X, \infty\}$.

Theorem 2.3 (Nall)

Let X_i ($i \in \mathbb{N}$) be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function for each $i \in \mathbb{N}$ such that one of the following conditions (1) and (2) is satisfied;
(1) $\dim f_{i,i+1}(x) = 0$ *for each* $i \in \mathbb{N}$ *and* $x \in X_{i+1}$, *i.e.,* $D_1(f_{i,i+1}) = \emptyset$.
(2) $\dim f_{i,i+1}^{-1}(x) = 0$ *for each* $i \in \mathbb{N}$ *and* $x \in X_i$, *i.e.,* $D_1(f_{i,i+1}^{-1}) = \emptyset$.
Then $\dim \varprojlim\{X_i, f_{i,i+1}\} \leq \sup\{\dim X_i \mid i \in \mathbb{N}\}$.

Theorem 2.4 (Ingram)

Let X_i ($i \in \mathbb{N}$) be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function for each $i \in \mathbb{N}$. If for each $i > 0$, Z_i is a closed 0-dimensional subset of X_i such that $g_{i,i+1} = f_{i,i+1}|(X_{i+1} - Z_{i+1})$ is a mapping and $f_{i,j}^{-1}(Z_i)$ is 0-dimensional for each $i \geq 2$ and $j > i$, then $\dim \varprojlim\{X_i, f_{i,i+1}\} \leq \sup\{\dim X_i \mid i \in \mathbb{N}\}$.

To evaluate dimension of generalized inverse limits, we need the following notations.

For a function $f : X \to 2^Y$, we put

$$D_1(f) = \{x \in X |\ \dim f(x) \geq 1\},\ D_1(f^{-1}) = \{y \in Y |\ \dim f^{-1}(y) \geq 1\},$$

where $f^{-1}(B) = \{x \in X |\ f(x) \cap B \neq \emptyset\}$ for a subset B of Y.

Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function for each $i \in \mathbb{N}$.
Let $y \in X_n$ and $x \in X_{n'}$ $(n \leq n')$. We consider the following conditions:

$$\boxed{y \leftarrow x} : \; y \in f_{n,n'}(x)$$

$$\boxed{x \lhd} : \; x \in D_1(f^{-1}_{n',n'+1})$$

$$\boxed{\rhd y} : \; n \geq 2 \text{ and } y \in D_1(f_{n-1,n})$$

Also, let $x \in X_m$ and $y \in X_{m'}$ $(m + 2 \leq m')$. We consider the following condition:

$$\boxed{x \longleftarrow\!\!\prec \rhd y} : \; y \in D_1(f_{m'-1,m'}) \text{ and } \dim[f^{-1}_{m,m'-1}(x) \cap f_{m'-1,m'}(y)] \geq 1$$

In particular, we also consider the following condition:

$$\boxed{x \Diamond y} : \; m' = m + 2, x \in D_1(f^{-1}_{m,m+1}), y \in D_1(f_{m+1,m+2}) \text{ and}$$

$$\dim[f^{-1}_{m,m+1}(x) \cap f_{m+1,m+2}(y)] \geq 1.$$

For each $x_n \in X_n$ with $x_n \in D_1(f_{n,n+1}^{-1})$, we consider the following sequence:

$$\triangleright y_{m_1} \leftarrowtail \triangleright y_{m_2} \leftarrowtail \triangleright y_{m_3} \leftarrowtail \cdots \leftarrowtail \triangleright y_{m_{k-1}} \leftarrowtail \triangleright y_{m_k} \leftarrow x_n \triangleleft,$$

where $2 \leq m_1, m_k \leq n$, $m_i + 2 \leq m_{i+1}$ $(i = 1, 2, ..., k-1)$ and $y_{m_i} \in X_{m_i}$ $(i = 1, 2, ..., k)$. In this case, we say that the sequence $\{y_{m_i}, x_n |\ 1 \leq i \leq k\}$ is *an expand-contract sequence* in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ with length k. For any expand-contract sequence

$$S : \triangleright y_{m_1} \leftarrowtail \triangleright y_{m_2} \leftarrowtail \triangleright \cdots \leftarrowtail \triangleright y_{m_{k-1}} \leftarrowtail \triangleright y_{m_k} \leftarrow x_n \triangleleft,$$

we put $d(S) = \sum_{i=1}^{k} \dim f_{m_i-1,m_i}(y_{m_i})$. We define the index $\tilde{J}(\{X_i, f_{i,i+1}\})$ as follows.

$$\tilde{J}(\{X_i, f_{i,i+1}\})$$

$$= \sup\{d(S) \mid S \text{ is an expand-contract sequence in } \{X_i, f_i\}_{i=1}^{\infty}\}.$$

The following is the main theorem of my talk.

Theorem 2.5

Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function for each $i \in \mathbb{N}$. Suppose that $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$. *Then*

$$\dim \varprojlim \{X_i, f_{i,i+1}\} \leq \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Theorem 2.6

Let X_i $(i \in \mathbb{N})$ be a sequence of 1-dimensional compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be a surjective upper semi-continuous function for each $i \in \mathbb{N}$. Suppose that each $i \geq 2$, Z_i is a 0-dimensional closed subset of X_i such that $f_{i,i+1}|X_{i+1} - Z_{i+1} : (X_{i+1} - Z_{i+1}) \to X_i$ is a mapping for each $x \in X_{i+1} - Z_{i+1}$ and $i \in \mathbb{N}$. Then

$$\tilde{J}(\{X_i, f_{i,i+1}\}) \leq \dim \varprojlim\{X_i, f_{i,i+1}\} \leq \tilde{J}(\{X_i, f_{i,i+1}\}) + 1.$$

Moreover, if there is an expand-contract sequence

$$\rhd y_{m_1} \leftarrowtail \rhd y_{m_2} \leftarrowtail \rhd \cdots \leftarrowtail \rhd y_{m_{k-1}} \leftarrowtail \rhd y_{m_k} \leftarrow x_n \lhd$$

in $\{X_i, f_{i,i+1}\}$ with length $\tilde{J}(\{X_i, f_{i,i+1}\}) = k$ such that $\dim \pi_n^{-1}(x_n) > 0$, then $\dim \varprojlim\{X_i, f_{i,i+1}\} = \tilde{J}(\{X_i, f_{i,i+1}\}) + 1$, where $\pi_n : \varprojlim\{X_i, f_{i,i+1}\}_{i \geq n} \to X_n$ is the projection defined by $\pi_n(x_n, x_{n+1}, \cdots) = x_n$.

Now, we will define another index $\tilde{I}(\{X_i, f_{i,i+1}\})$ as follows. Let $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ be an inverse sequence with set-valued functions. Also, let $x \in X_m$ and $y \in X_{m'}$ $(m+2 \leq m')$. We consider the following condition:

$$\boxed{x \lhd \succ y} : \ x \in D_1(f_{m,m+1}^{-1}) \text{ and } \dim[f_{m,m+1}^{-1}(x) \cap f_{m+1,m'-1}(y)] \geq 1$$

Note that $x \Diamond y$ implies $x \prec \rhd y$ and $x \lhd \succ y$.

For each $x_n \in X_n$ with $x_n \in D_1(f^{-1}_{n,n+1})$, we consider the following sequence:

$$\rhd x_n \leftarrow y_{m_1} \lhd \succ y_{m_2} \lhd \succ y_{m_3} \lhd \succ \cdots \lhd \succ y_{m_{k-1}} \lhd \succ y_{m_k} \lhd$$

where $n \leq m_1, m_i + 2 \leq m_{i+1}$ $(i = 1, 2, ..., k-1)$ and $y_{m_i} \in X_{m_i}$ $(i = 1, 2, ..., k)$.

In this case, we say that the sequence $(x_n, y_{m_1}, y_{m_2}, \cdots, y_{m_k})$ is an inverse expand-contract sequence in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ with length k. Note that a sequence $(x_n, y_{m_1}, y_{m_2}, \cdots, y_{m_k})$ is an inverse expand-contract sequence in the inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ if and only if the sequence $(y_{m_k}, y_{m_{k-1}}, \cdots, y_{m_1}, x_n)$ is an expand-contract sequence in the direct sequence $\{X_i, f_{i,i+1}^{-1}\}_{i=1}^{\infty}$.

For any inverse expand-contract sequence

$$S : \rhd x_n \leftarrow y_{m_1} \lhd \succ y_{m_2} \lhd \succ y_{m_3} \lhd \succ \cdots \lhd \succ y_{m_{k-1}} \lhd \succ y_{m_k} \lhd$$

we put $d(S) = \sum_{i=1}^{k} \dim f_{m_i, m_i+1}^{-1}(y_{m_i})$. We define the index $\tilde{I}(\{X_i, f_{i,i+1}\})$ as follows.

$$\tilde{I}(\{X_i, f_{i,i+1}\})$$

$$= \sup\{d(S) \mid S \text{ is an inverse expand-contract sequence in} \{X_i, f_{i,i+1}\}\}.$$

If there is no inverse expand-contract sequence in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$, we put $\tilde{I}(\{X_i, f_{i,i+1}\}) = 0$. In general,

$$\tilde{J}(\{X_i, f_{i,i+1}\}) \neq \tilde{I}(\{X_i, f_{i,i+1}\}).$$

Theorem 2.7

Let X_i $(i \in \mathbb{N})$ *be a sequence of compacta and let* $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ *be an upper semi-continuous function for each* $i \in \mathbb{N}$*. Suppose that* $\dim D_1(f_{i,i+1}^{-1}) \leq 0$ $(i \in \mathbb{N})$*. Then*

$$\dim \varprojlim \{X_i, f_{i,i+1}\} \leq \tilde{I}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Examples

Example 1. Let $n \in \mathbb{N}$ with $n \geq 2$ and let $f : I \to C(I)$ be the surjective upper semi-continuous function defined by $f(x) = 0$ $(x \in [0, 1/n))$ and for $1 \leq i \leq n-1$, $f(i/n) = [(i-1)/n, i/n], f(x) = i/n$ $(x \in (i/n, (i+1)/n)), f(1) = [(n-1)/n, 1]$. Then

$$\triangleright 1/n \Diamond 2/n \Diamond \cdots \Diamond (n-1)/n \triangleleft$$

is a maximal expand-contract sequence and hence $\tilde{J}(\{I, f\}) = n - 1$. In fact, we see that $\varprojlim\{I, f\}$ is an n-dimensional stepwise polyhedron.

Example 2. There is an inverse sequence $\{I_i, f_{i,i+1}\}$ of intervals with surjective upper semi-continuous functions such that $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$ and

$$0 = \dim \varprojlim \{I_i, f_{i,i+1}\} \neq \tilde{J}(\{I_i, f_{i,i+1}\}) + 1 = 2.$$

Let C be a Cantor set in $[0, 1/2]$. Let $u : C \to [0, 1/2]$ be a surjective map. Consider the following surjective upper semi-continuous functions $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ $(i \in \mathbb{N})$:

(1) $f_{1,2}(x) = u^{-1}(x)$ $(x \in [0, 1/2])$ and $f_{1,2}|[1/2, 1] : [1/2, 1] \to I$ is an onto map.

(2) $f_{2,3}(x) = x$ $(x \in [0, 1/2))$, $f_{2,3}(1/2) = [0, 1/2]$, $f_{2,3}(x) = x$ $(x \in (1/2, 1])$.

(3) $f_{3,4}(x) = x$ $(x \in [0, 1/2))$, $f_{3,4}(x) = \{1/2, x\}$ $(x \in [1/2, 1])$.

Also, we will construct $f_{i,i+1}$ $(i \geq 4)$ as follows. For any $\epsilon > 0$, we can construct a surjective upper semi-continuous function $f_\epsilon : [1/2, 1] \to 2^{[1/2,1]}$ such that for some sequence $1/2 = t_0 < t_1 < t_2 < \cdots < t_{s-1} < t_s = 1$,

(a) $f_\epsilon(1/2) = 1/2, f_\epsilon(1) = 1$,

(b) $f_\epsilon|(t_i, t_{i+1})$ $(i = 1, 2, ..., t_{s-1})$ is an injective map and $f_\epsilon([1/2, 1]) = [1/2, 1]$,

(c) $f_\epsilon(t_i)$ is two point set for $i = 1, 2, ..., t_{s-1}$ and each diameter of $G(f_\epsilon|(t_i, t_{i+1}))$ $(\subset G(f_\epsilon))$ is less than ϵ.

By use of maps $f_\epsilon : [1/2, 1] \to 2^{[1/2,1]}$ for sufficiently small $\epsilon > 0$ and by induction on i (≥ 4) we can construct surjective upper-semi continuous functions $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ such that $f_{i,i+1}|[0, 1/2] = id$ and $\dim \varprojlim\{[1/2, 1], f_{i,i+1}|[1/2, 1]\}_{i=4}^{\infty} = 0$. Note that

$$\triangleright x_3 = 1/2 \leftarrow x_3 = 1/2 \triangleleft \ \ (x_3 \in I_3).$$

In fact, $J(\{I_i, f_{i,i+1}\}) = 1$. Since $\dim \varprojlim\{[1/2, 1], f_{i,i+1}|[1/2, 1]\}_{i=4}^{\infty} = 0$, we see that $\dim \pi_3^{-1}(x_3) = 0$ and hence $\dim \varprojlim\{I_i, f_{i,i+1}\} = 0$.