Baire one functions depending on finitely many coordinates

Olena Karlova

Chernivtsi National University

Definitions and notations

$P = \prod_{n=1}^{\infty} X_n,\ a = (a_n), x = (x_n) \in P$

$$p_n(x) = (x_1, \dots, x_n, a_{n+1}, a_{n+2}, \dots)$$

- $A \subseteq P$ ***depends on finitely many coordinates*** $\equiv \exists n \in \mathbb{N}$ $\forall x \in A\ \forall y \in P$

$$p_n(x) = p_n(y) \implies y \in A.$$

- A map $f : X \to Y$ defined on a subspace $X \subseteq P$ is ***finitely determined*** $\equiv \exists n \in \mathbb{N}\ \forall x, y \in X$

$$p_n(x) = p_n(y) \implies f(x) = f(y).$$

- $\mathrm{CF}(X, Y)$ is the set of all continuous finitely determined maps between X and Y; $\mathrm{CF}(X) = \mathrm{CF}(X, \mathbb{R})$.

Vladimir Bykov's results

V.Bykov, *On Baire class one functions on a product space*, Topol. Appl. 199 (2016) 55–62.

Theorem

Let X be a subspace of a product $P = \prod_{n=1}^{\infty} X_n$ of a sequence of metric spaces X_n. Then

1. every Baire class one function $f : X \to \mathbb{R}$ is the pointwise limit of a sequence of functions from $\mathrm{CF}(X)$;
2. a lower semicontinuous function $f : X \to \mathbb{R}$ is the pointwise limit of an increasing sequence of functions from $\mathrm{CF}(X) \Leftrightarrow f$ has a minorant in $\mathrm{CF}(X)$.

Vladimir Bykov's questions

V.Bykov, *On Baire class one functions on a product space*, Topol. Appl. 199 (2016) 55–62.

Questions

Let X be a subspace of a product $P = \prod_{n=1}^{\infty} X_n$ of a sequence of ~~metric~~ spaces X_n. Is

1. every Baire class one function $f : X \to \mathbb{R}$ a pointwise limit of a sequence of functions from $\mathrm{CF}(X)$ for **completely regular** X?
2. a lower semicontinuous function $f : X \to \mathbb{R}$ a pointwise limit of an increasing sequence of functions from $\mathrm{CF}(X)$ for **perfectly normal** X?

Positive answers

- A map $f : X \to Y$ is ***F_σ-measurable*** $\equiv f^{-1}(V)$ is F_σ in X for any open $V \subseteq Y$.

Baire one $\Longrightarrow$ F_σ-measurable

Theorem 1

Let $P = \prod_{n=1}^{\infty} X_n$ be a completely regular space, $X \subseteq P$ and Y be a path-connected space. If

1. P is perfectly normal, or
2. X is Lindelöf,

then every F_σ-measurable function $f : X \to Y$ with countable discrete image $f(X)$ is a pointwise limit of a sequence of functions from $\mathrm{CF}(X, Y)$.

Positive answers

For $f, g : X \to Y$ we write $(f\Delta g)(x) = (f(x), g(x))$ for all $x \in X$.

A family $\mathscr{F}$ of maps between X and Y is called

- ***Δ-closed*** $\equiv h \circ (f\Delta g) \in \mathscr{F}$ for any $f, g \in \mathscr{F}$ and any continuous map $h : Y^2 \to Y$.

$\mathrm{B}_1(X, Y)$ and $\mathrm{CF}(X, Y)$ are Δ-closed

Positive answers

A metric space (Y, d) is called

- ***an R-space*** $\equiv \forall \varepsilon > 0 \; \exists r_\varepsilon \in C(Y \times Y, Y)$

$$d(y,z) \le \varepsilon \implies r_\varepsilon(y,z) = y, \tag{1}$$

$$d(r_\varepsilon(y,z), z) \le \varepsilon \tag{2}$$

for all $y, z \in Y$.

Any convex subset Y of a normed space is an R-space

Positive answers

Theorem 2

Let $P = \prod_{n=1}^{\infty} X_n$ be a completely regular space, $X \subseteq P$ and Y be a path-connected metric separable R-space. If

1. P is perfectly normal, or
2. X is Lindelöf,

then

① $F_\sigma(X, Y) = \mathrm{B}_1(X, Y) = \overline{\mathrm{CF}(X, Y)}^{\mathrm{p}}$.

If, moreover, X is perfectly normal, then

② any lower semicontinuous function $f : X \to [0, +\infty)$ is a pointwise limit of an increasing sequence of functions from $\mathrm{CF}(X, [0, +\infty))$.

Pseudocompact case

Theorem 1

Let $P = \prod_{n=1}^{\infty} X_n$ be a pseudocompact space and Y be a path-connected separable metric R-space. Then

$$\mathrm{B}_1(P, Y) = \overline{\mathrm{CF}(P, Y)}^{\mathrm{p}}.$$

Pseudocompact case

Theorem 1

Let $P = \prod_{n=1}^{\infty} X_n$ be a pseudocompact space and Y be a path-connected separable metric R-space. Then

$$\mathrm{B}_1(P, Y) = \overline{\mathrm{CF}(P, Y)}^{\mathrm{p}}.$$

Question

Let $X \subseteq \prod_{n=1}^{\infty} X_n$ be a pseudocompact subspace of a product of completely regular spaces X_n and $f : X \to \mathbb{R}$ be a Baire one function. Does there exist a sequence of functions from $\mathrm{CF}(X)$ which is pointwisely convergent to f on X?

Negative answer

Theorem 3

There exist a sequence $(X_n)_{n=1}^{\infty}$ of Lindelöf spaces X_n and a function $f \in \mathrm{B}_1(\prod_{n=1}^{\infty} X_n, \mathbb{R})$ such that

1. every finite product $Y_n = \prod_{k=1}^{n} X_k$ is Lindelöf;
2. f is not a pointwise limit of any sequence $(f_n)_{n=1}^{\infty}$ of functions from $\mathrm{CF}(\prod_{n=1}^{\infty} X_n)$.