SELECTION PRINCIPLES IN UNIFORM TOPOLOGY

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Abstract

The theory of uniform spaces and uniformly continuous mappings is one of the main directions of modern topology, which is being intensively developed at the present time, and uniform spaces have long occupied an important place both in topology itself and in its applications.

Although the theory of uniform spaces and uniformly continuous mappings is independent, it is closely related to the theory of topological spaces and continuous mappings, and there is a deep analogy between them.

At a seminar on topology at the Charles University (Prague) Z. Frolik posed the following general problem: *To find and to investigate uniform analogues of the most important classes of topological spaces and continuous mappings (uniformization problem).*

So, the problem of finding and studying of uniform analogues of classes of topological spaces is topical. Uniform analogues of classical topological concepts like Menger, Hurewicz and Rothberger properties were first introduced by L.D.R. Kocinac [3]. In this talk, uniform Menger, uniform Hurewicz and uniform Rothberger spaces are studied. The most important characteristics of these properties are established.

Basic notions on selection principles in topological spaces and uniform spaces.

Let A and B be collections of open covers of a topological space X.
The symbol $S_1(A, B)$ denotes the selection hypothesis that for each sequence $\{\alpha_n\}$ of elements of A there exists a sequence $\{U_n\}$ such that for each A, $U_n \in \alpha_n$ and $\{U_n\} \in B$ [3]. And the symbol $S_{fin}(A, B)$ is the selection hypothesis that for each sequence $\{\alpha_n\}$ of elements of A there exists a sequence $\{V_n\}$ such that for each A, V_n is a finite subset of $\{U_n\}$ and $\bigcup_{n \in N} V_n$ is an element of B [3].
If O denotes the collection of all open covers of a space X, then X is said to have the Menger property [5] (resp. the Rothberger property [6], if the selection hypothesis exists a sequence $S_{fin}(O, O)$ (resp. $S_1(O, O)$) is true for X.
In [2], W. Hurewicz introduced a covering property for a topological space X, called now the Hurewicz property, in this way: for each sequence $\{\alpha_n\}$ of open covers of X there is a sequence $\{V_n\}$ of finite sets such that for each n, $V_n \subset \alpha_n$, and for each $x \in X$, for all but finitely many n, $x \in \bigcup V_n$.

Definitions

Let (X, U) be a uniform space. A uniform space (X, U) is called *uniformly Menger space* or has the *uniform Menger property*, if for each sequence $\{\alpha_n\} \subset U$ there is a sequence $\{\beta_n\}$ such that for each $n \in N$ β_n is a finite subset of α_n and $\bigcup_{n \in N} \beta_n$ is a cover of X, [3]; a uniform space (X, U) is called *uniformly Hurewicz space* or has the *uniform Hurewicz property*, if for each sequence $\{\alpha_n\} \subset U$ there is a sequence $\{\beta_n\}$, such that each β_n is a finite subset of α_n and for each $x \in X$ we have $x \in \bigcup \beta_n$ for all but finitely many n, [3]; a uniform space (X, U) is called *uniformly Rothberger space* or has the *uniform Rothberger property*, if for each sequence $\{\alpha_n\} \subset U$ there is a sequence $\{D_n\}$ such that for each $n \in N$ $D_n \in \alpha_n$ and $\bigcup_{n \in N} D_n = X$, [3].

Denotations

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}.\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \bigcup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \bigcap H \neq \emptyset\}$, $H \subset X$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, the symbol $\alpha * \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$such that $\alpha(A) \subset B$. For the uniformity U by τ_U we denote the topology generated by the uniformity and symbol U_X means the universal uniformity.

Definitions

A uniform space (X, U) is called:

1. *precompact*, if the uniformity U has a base consisting of countable covers [1];
2. *pre-Lindelöf* or ($=\aleph_0$-bounded), if the uniformity U has a base consisting of countable covers [1];
3. *uniformly locally compact*, if the uniformity of U contains a uniform covering consisting is compact sets [1].

A uniformly continuous mapping $f : (X, U) \to (Y, V)$ between uniform spaces (X, U) and (Y, V) is called:

1. *precompact*, if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and finite uniform covering $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [1];
2. *uniformly perfect*, if it is both precompact and perfect [1].

Throughout this talk all uniform spaces are assumed to be Hausdorff, topological spaces are assumed to be Tychonoff and mappings are uniformly continuous.

Some Properties of Uniformly Menger, Uniformly Hurewicz and Uniformly Rothberger Spaces. Main Results.

The theory of selection principles has been developing intensively. Finding of uniform analogues of the basic concepts and statements of the theory of selection principles is an actual task of the uniform topology.
Let (X, U) be uniform space. Any precompact uniform space (X, U) is a uniform Hurewicz space and any uniform Hurewicz space is uniformly Menger space. Any uniform Rothberger space (X, U) is uniformly Menger space and any uniformly Menger space is pre-Lindelöf. Hence, it follows that the classes of uniformly Menger, uniformly Hurewicz and uniformly Rothberger spaces are different.

If X is a Menger (resp. Hurewicz, Rothberger) space, then (X, U) is uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) space and the converse is not always true. However, the following assertion is true.

Proposition 1.

A Tychonoff space X is a Menger (resp. Hurewicz, Rothberger) space if and only if the uniform space (X, U_X), where U_X is the universal uniformity, is a uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) space.

Main Results

The following two theorems show that the uniformly Menger, uniformly Hurewicz and uniformly Rothberger properties are preserved when passing to any subspaces and a finite disjunct sum of uniform spaces.

Theorem 1.

Every subspace of a uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) space is uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger).

Theorem 2.

A finite discrete sum $(X, U) = \{(X_i, U_i) i = 1, 2, ..., m\}$ uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) spaces $(X_i, U_i), i = 1, 2, ..., m$ is uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger).

Main Results

The following theorem shows that the uniformly Hurewicz and uniformly Rothberger properties are closed under the product operation.

Theorem 3.

The product $(X \times Y, U \times V)$ of uniformly Hurewicz (resp. uniformly Rothberger) spaces (X, U) and (Y, V) is uniformly Hurewicz (resp. uniformly Rothberger).

Theorem 4.

The product $(X \times Y, U \times V)$ of a uniformly Menger space (X, U) and precompact uniform space (Y, V) is a uniformly Menger space.

Main Results

Corollary 1.

The product of a uniformly Menger space and compact uniform space is a uniformly Menger space.

Theorem 5.

The completion of a uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) space is a uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) space.

Remind, a system α of (X, U) is called *co-cover*, if $\alpha \bigcap F \neq \emptyset$ for every free Cauchy filter F of (X, U), [4].

Main Results

Theorem 6.

The remainder $(\hat{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of a uniform space (X, U) is

1. uniformly Menger space;
2. uniformly Hurewicz space;
3. uniformly Rothberger space

if and only if

(a1) for any sequence $\{\alpha_n\} \subset U$ there exists a sequence of $\{\alpha_n^0\}$ finite subfamilies such that $\bigcup_{n\in N} \alpha_n^0$ is a co-cover of the uniform space (X, U);

(a2) for any sequence $\{\alpha_n\} \subset U$ there exists a sequence of $\{\alpha_n^0\}$ finite subfamilies and for each free Cauchy filter F of (X, U) we have $\bigcup \alpha_n^0 \in F$ for all but finitely many n;

(a3) for any sequence $\{\alpha_n\} \subset U$ there exists a sequence $\{D_n\}$, such that $\{D_n\}$ is a co-cover of the uniform space (X, U).

Main Results

The following theorem shows that under precompact mappings, the three properties are preserved in both directions.

Theorem 7.

Let $f : (X, U) \to (Y, V)$ be a precompact mapping between uniform spaces (X, U) and (Y, V). Then uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) properties is preserved both in the image and the preimage direction.

Corollary 2.

Let $f : (X, U) \to (Y, V)$ be a uniformly perfect mapping between uniform spaces (X, U) and (Y, V). Then uniformly Menger (resp. uniformly Hurewicz, uniformly Rothberger) properties is preserved both in the image and the preimage direction.

Main Results

Proposition 2.

The space of real numbers R with natural uniformity is a uniformly Menger space.

Corollary 3.

The space of the rational numbers Q induced from the uniformity U_R is a uniformly Menger space. The unit interval $(0, 1)$ induced from the uniformity U_R is also uniformly Menger space.

Proposition 3.

A uniformly locally compact space (X, U) is a uniformly Menger space if and only if it is a pre-Lindelöf.

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Thanks for attention.