Measuring noncompactness and discontinuity

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Plan

Compactness in metric spaces, norm-compactness

- Measuring non-compactness in a metric space
- Norm-compactness and continuity of operators

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References

[CKS 2012] B.Cascales, O.Kalenda and J.Spurný: A quantitative version of James' compactness theorem, Proc. Edinburgh Math. Soc., II. Ser. 55 (2012), no. 2, 369-386.

[KKS 2013] M.Kačena, O.Kalenda and J.Spurný: Quantitative Dunford-Pettis property, Advances in Math. 234 (2013), 488-527.

[KS 2012] O.Kalenda and J.Spurný: Quantification of the reciprocal Dunford-Pettis property, Studia Math. 210 (2012), no. 3, 261-278.

. . . and some recent observations

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Compactness in metric spaces, norm-compactness

Measuring non-compactness in a metric space

Norm-compactness and continuity of operators

Weak non-compactness

Two approaches to weak noncompactness

Comparison of the two approaches

Weak compactness and continuity

Mackey non-compactness

Measuring Mackey non-compactness

Mackey compactness and continuity

Application: Dunford-Pettis property

Measuring non-compactness in a metric space

Theorem

Let (X, d) be a complete metric space and $A \subset X$. TFAE:

- A is relatively compact.
- A is totally bounded.
- Any sequence in A has a subsequence converging in X.

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- $\chi(A) = 0 \Leftrightarrow A$ is totally bounded
- $\chi(A) \leq \chi_0(A) \leq \beta(A) \leq \alpha(A) \leq 2\chi(A)$

Compactness and continuity of operators

Measuring non-compactness of an operator

$T : X \to Y$ . . . a bounded operator between Banach spaces.

- T is compact iff TB_X is relatively compact.

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- $T^*|_{B_{Y^*}}$ is w*-to-norm continuous iff $\forall (y^*_\tau) \subset B_{Y^*}$: (y^*_τ) w*-convergent $\Rightarrow$ $(T^* y^*_\tau)$ norm-convergent

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Compactness and continuity – quantitative relation

$$\frac{1}{2} \operatorname{cont}_{w^* \to \|\cdot\|}(T^*) \le \chi(T) \le \operatorname{cont}_{w^* \to \|\cdot\|}(T^*)$$

Plan

Measuring weak non-compactness - de Blasi approach

X ... a Banach space
A ... a bounded subset of X.

Recall:

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Measuring how much a set sticks out of another one

$$\widehat{\mathrm{d}}(A, B) = \sup\{\operatorname{dist}(a, B); a \in A\}$$

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Hausdorff measure of noncompactness reformulated

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$$\chi(A) = \inf\{\widehat{\mathrm{d}}(A, F); F \subset X \text{ finite}\}$$

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De Blasi measure of weak noncompactness

- $\omega(A)=\inf\{\widehat{\mathrm{d}}(A,K);K\subset X\text{ weakly compact}\}$

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- $\omega(A) = \inf\{\widehat{\mathrm{d}}(A,K); K \subset X \text{ weakly compact}\}$
- [de Blasi 1977] $\omega(A) = 0 \Leftrightarrow A$ is relatively weakly compact

Other measures of weak noncompactness

Let X be a Banach space and $A \subset X$ a bounded set. TFAE

- A is relatively weakly compact.
- [Banach-Alaoglu] $\overline{A}^{w^*} \subset X$
- [Eberlein-Grothendieck] $\lim_i \lim_j x_i^*(x_j) = \lim_j \lim_i x_i^*(x_j)$ whenever $(x_j) \subset A$, $(x_i^*) \subset B_{X^*}$ and all limits exist.
- [Eberlein-Šmulyan] Any $(x_n) \subset A$ has a w-cluster point in X.
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- [James] Any $x^* \in X^*$ attains its max on $\overline{A}^w$.

$$\mathrm{Ja}(A) = \inf\{r > 0; \forall x^* \in E^* \, \exists x^{**} \in \overline{A}^{w^*} : \\ x^{**}(x^*) = \sup x^*(A) \;\&\; \mathrm{dist}(x^{**}, X) \le r\}$$

Quantitative characterizations of weak compactness

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Let $X = C(K)$. Are $\omega(A)$ and $\mathrm{wk}(A)$ equivalent for bounded subsets of X?

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Let $X = C(K)$. Are $\omega(A)$ and $\mathrm{wk}(A)$ equivalent for bounded subsets of X? Is it true at least for $K = [0, 1]$?

Comparison of measures of weak non-compactness II

Theorem

Let X be a Banach space.

- X is WCG iff $\forall \varepsilon > 0\, \exists\, (A_n)_{n=1}^{\infty}$ a cover of X $\forall n \in \mathbb{N} : \omega(A_n) < \varepsilon$.

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[An exercise]

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- X is a subspace of WCG iff
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Comparison of measures of weak non-compactness II

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Remark

If ω and wk are equivalent in $C(K)$ spaces, it easily follows that Eberlein compact spaces are preserved by continuous images.

Weak compactness and continuity

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[Gantmacher 1940]

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$T^*|_{B_{X^*}}$ w*-to-w continuous

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[Peralta, Villanueva, Maitland Wright and Ylinen 2007]

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- $\omega(T)$ and $\omega(T^*)$ are incomparable.

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Corollary

$\frac{1}{4}\,\mathrm{cont}_{w^*\to w}(T^*) \le \mathrm{wk}(T^*) \le 2\,\mathrm{wk}(T) \le 4\,\mathrm{cont}_{w^*\to w}(T^*)$

Plan

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Question

Are the quantities χ_m and ω_m equivalent in any dual space?

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Remark

$\frac{1}{2}\chi_m(A) \le \omega_m(A)$ holds always.

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Let X be a Banach space. The following assertions are equivalent to the Dunford-Pettis property of X.

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Quantitative strengthening of DPP [KKS 2013]

- X has direct qDPP if
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- X has dual qDPP if
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Thank you for your attention.