

Measuring noncompactness and discontinuity

Ondřej F.K. Kalenda

Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University in Prague

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Compactness in metric spaces, norm-compactness

Measuring non-compactness in a metric space

Norm-compactness and continuity of operators

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- Two approaches to weak noncompactness

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Application: Dunford-Pettis property

- [CKS 2012] B.Cascales, O.Kalenda and J.Spurný: A quantitative version of James' compactness theorem, Proc. Edinburgh Math. Soc., II. Ser. 55 (2012), no. 2, 369-386.
- [KKS 2013] M.Kačena, O.Kalenda and J.Spurný: Quantitative Dunford-Pettis property, Advances in Math. 234 (2013), 488-527.
- [KS 2012] O.Kalenda and J.Spurný: Quantification of the reciprocal Dunford-Pettis property, Studia Math. 210 (2012), no. 3, 261-278.

... and some recent observations

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Theorem

Let (X, d) be a complete metric space and $A \subset X$. TFAE:

- ▶ A is relatively compact.
- ▶ A is totally bounded.
- ▶ Any sequence in A has a subsequence converging in X .

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Let (X, d) be a ~~complete~~ metric space and $A \subset X$. TFAE:

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$T : X \rightarrow Y$... a bounded operator between Banach spaces.

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Compactness and continuity – quantitative relation

$$\frac{1}{2} \text{cont}_{w^* \rightarrow \|\cdot\|} (T^*) \leq \chi(T) \leq \text{cont}_{w^* \rightarrow \|\cdot\|} (T^*)$$

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X ... a Banach space

A ... a bounded subset of X .

Recall:

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$$\widehat{d}(A, B) = \sup\{\text{dist}(a, B); a \in A\}$$

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- ▶ $\omega(A) = \inf\{\widehat{d}(A, K); K \subset X \text{ weakly compact}\}$
- ▶ [de Blasi 1977] $\omega(A) = 0 \Leftrightarrow A$ is relatively weakly compact

Other measures of weak noncompactness

Let X be a Banach space and $A \subset X$ a bounded set. TFAE

- ▶ A is relatively weakly compact.
- ▶ [Banach-Alaoglu] $\overline{A}^{w*} \subset X$
- ▶ [Eberlein-Grothendieck] $\lim_i \lim_j x_i^*(x_j) = \lim_j \lim_i x_i^*(x_j)$ whenever $(x_j) \subset A$, $(x_i^*) \subset B_{X^*}$ and all limits exist.
- ▶ [Eberlein-Šmuljan] Any $(x_n) \subset A$ has a w-cluster point in X .
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$$\text{Ja}(A) = \inf\{r > 0; \forall x^* \in E^* \exists x^{**} \in \overline{A}^{w*} :$$

$$x^{**}(x^*) = \sup x^*(A) \text{ \& } \text{dist}(x^{**}, X) \leq r\}$$

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- ▶ James theorem [CKS 2012]

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Question

Let $X = C(K)$. Are $\omega(A)$ and $\text{wk}(A)$ equivalent for bounded subsets of X ?

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 - ▶ [KKS 2013] $X = c_0(\Gamma)$
 - ▶ [KKS 2013] $X = L^1(\mu)$

$$\text{wk}(A) = \omega(A) = \inf \left\{ \sup_{f \in A} \int (|f| - c\chi_E)^+ d\mu : c > 0, \mu(E) < +\infty \right\}$$

- ▶ [in preparation] $X = N(H)$ or $X = K(H)$

Question

Let $X = C(K)$. Are $\omega(A)$ and $\text{wk}(A)$ equivalent for bounded subsets of X ? **Is it true at least for $K = [0, 1]$?**

Theorem

Let X be a Banach space.

- ▶ X is **WCG** iff

$\forall \varepsilon > 0 \exists (A_n)_{n=1}^{\infty}$ a cover of $X \forall n \in \mathbb{N} : \omega(A_n) < \varepsilon$.

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Remark

If ω and \mathbf{wk} are equivalent in $C(K)$ spaces, it easily follows that Eberlein compact spaces are preserved by continuous images.

Weak compactness and continuity

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[Gantmacher 1940]

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$$T \text{ weakly compact} \quad \Leftrightarrow \quad \begin{array}{l} T^*|_{B_{X^*}} \text{ w* -to-w continuous} \\ \Updownarrow \\ T^* \text{ weakly compact} \end{array}$$

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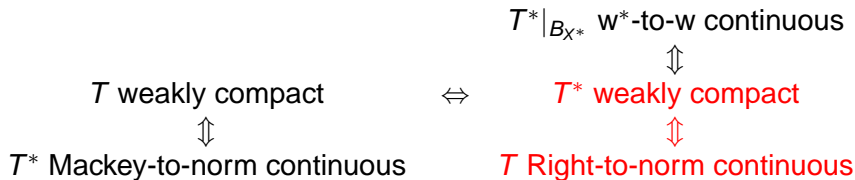
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$\rho(X, X^*) = \mu(X^{**}, X^*)|_X$

Measuring discontinuity

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- ▶ $\omega(T)$ and $\omega(T^*)$ are incomparable.

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Corollary

$$\frac{1}{4} \text{cont}_{w^* \rightarrow w} (T^*) \leq \text{wk}(T^*) \leq 2 \text{wk}(T) \leq 4 \text{cont}_{w^* \rightarrow w} (T^*)$$

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Are the quantities χ_m and ω_m equivalent in any dual space?

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Remark

$\frac{1}{2}\chi_m(A) \leq \omega_m(A)$ holds always.

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Quantitative strengthening of DPP [KKS 2013]

- ▶ X has **direct qDPP** if
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Thank you for your attention.