Hereditarily σ-metacompact function spaces.

Heikki Junnila

Hereditarily σ-metacompact function spaces.

Heikki Junnila

Hereditarily σ-metacompact function spaces.

Heikki Junnila

Introduction.

Let us consider the following problem:

Hereditarily σ-metacompact function spaces.

Heikki Junnila

Introduction.

Let us consider the following problem:

Problem 1 *Let P be a covering property. For what kind of compact spaces K does $C_p(K)$ satisfy P hereditarily?*

Hereditarily σ-metacompact function spaces.

Heikki Junnila

Introduction.

Let us consider the following problem:

Problem 1 *Let P be a covering property. For what kind of compact spaces K does $C_p(K)$ satisfy P hereditarily?*

The most important covering property,

Hereditarily σ-metacompact function spaces.

Heikki Junnila

Introduction.

Let us consider the following problem:

Problem 1 *Let P be a covering property. For what kind of compact spaces K does $C_p(K)$ satisfy P hereditarily?*

The most important covering property, beyond compactness,

Hereditarily σ-metacompact function spaces.

Heikki Junnila

Introduction.

Let us consider the following problem:

Problem 1 *Let P be a covering property. For what kind of compact spaces K does $C_p(K)$ satisfy P hereditarily?*

The most important covering property, beyond compactness, is paracompactness.

Hereditarily σ-metacompact function spaces.

Heikki Junnila

Introduction.

Let us consider the following problem:

Problem 1 *Let P be a covering property. For what kind of compact spaces K does $C_p(K)$ satisfy P hereditarily?*

The most important covering property, beyond compactness, is paracompactness.
For this property, the solution to the above problem is known.

-

Theorem *TFAE for a compact space K:*

Theorem *TFAE for a compact space K:*

A. $C_p(K)$ is hereditarily paracompact.

Theorem *TFAE for a compact space* K *:*

A. $C_p(K)$ *is hereditarily paracompact.*

B. $C_p(K)$ *is hereditarily Lindelöf.*

Theorem *TFAE for a compact space* K*:*

A. $C_p(K)$ *is hereditarily paracompact.*

B. $C_p(K)$ *is hereditarily Lindelöf.*

C. K *is hereditarily separable.*

Theorem *TFAE for a compact space K:*

A. $C_p(K)$ is hereditarily paracompact.

B. $C_p(K)$ is hereditarily Lindelöf.

C. K is hereditarily separable.

For various generalizations of paracompactness,

we do not have such clearcut solutions to Problem 1.

Theorem *TFAE for a compact space K:*

A. $C_p(K)$ *is hereditarily paracompact.*

B. $C_p(K)$ *is hereditarily Lindelöf.*

C. K *is hereditarily separable.*

For various generalizations of paracompactness,
we do not have such clearcut solutions to Problem 1.
For example, a paper by Tkachuk lists twenty problems
on metacompactness of C_p-spaces,

Theorem *TFAE for a compact space K:*

A. $C_p(K)$ is hereditarily paracompact.

B. $C_p(K)$ is hereditarily Lindelöf.

C. K is hereditarily separable.

For various generalizations of paracompactness,
we do not have such clearcut solutions to Problem 1.
For example, a paper by Tkachuk lists twenty problems
on metacompactness of C_p-spaces, and they seem to remain unsolved.

Theorem *TFAE for a compact space* K*:*

A. $C_p(K)$ *is hereditarily paracompact.*

B. $C_p(K)$ *is hereditarily Lindelöf.*

C. K *is hereditarily separable.*

For various generalizations of paracompactness,
we do not have such clearcut solutions to Problem 1.
For example, a paper by Tkachuk lists twenty problems
on metacompactness of C_p-spaces, and they seem to remain unsolved.

Here I will consider a generalization of metacompactness.

Theorem *TFAE for a compact space* K*:*

A. $C_p(K)$ *is hereditarily paracompact.*

B. $C_p(K)$ *is hereditarily Lindelöf.*

C. K *is hereditarily separable.*

For various generalizations of paracompactness,
we do not have such clearcut solutions to Problem 1.
For example, a paper by Tkachuk lists twenty problems
on metacompactness of C_p-spaces, and they seem to remain unsolved.

Here I will consider a generalization of metacompactness.
A space X is σ-*metacompact* if every open cover of X

Theorem *TFAE for a compact space K:*

A. $C_p(K)$ *is hereditarily paracompact.*

B. $C_p(K)$ *is hereditarily Lindelöf.*

C. K *is hereditarily separable.*

For various generalizations of paracompactness, we do not have such clearcut solutions to Problem 1. For example, a paper by Tkachuk lists twenty problems on metacompactness of C_p-spaces, and they seem to remain unsolved.

Here I will consider a generalization of metacompactness. A space X is σ-*metacompact* if every open cover of X has a σ-point-finite open refinement.

Theorem *TFAE for a compact space K:*

A. $C_p(K)$ is hereditarily paracompact.

B. $C_p(K)$ is hereditarily Lindelöf.

C. K is hereditarily separable.

For various generalizations of paracompactness,
we do not have such clearcut solutions to Problem 1.
For example, a paper by Tkachuk lists twenty problems
on metacompactness of C_p-spaces, and they seem to remain unsolved.

Here I will consider a generalization of metacompactness.
A space X is σ-*metacompact* if every open cover of X
has a σ-point-finite open refinement.
With this property, some solutions exist for Problem 1.

-

Gulko has shown that for every Corson compact space K, the space $C_p(K)$ is hereditarily σ-metacompact.

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,
then $C_p(\bar{T})$ is hereditarily σ-metacompact;

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,
then $C_p(\bar{T})$ is hereditarily σ-metacompact;
we also showed that $C_p(\beta\omega)$ is not σ-metacompact.

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,
then $C_p(\bar{T})$ is hereditarily σ-metacompact;
we also showed that $C_p(\beta\omega)$ is not σ-metacompact.

A space X is *countably metacompact*

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,
then $C_p(\bar{T})$ is hereditarily σ-metacompact;
we also showed that $C_p(\beta\omega)$ is not σ-metacompact.

A space X is *countably metacompact*
if every countable open cover of X has a point-finite open refinement.

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,
then $C_p(\bar{T})$ is hereditarily σ-metacompact;
we also showed that $C_p(\beta\omega)$ is not σ-metacompact.

A space X is *countably metacompact*
if every countable open cover of X has a point-finite open refinement.
X is metacompact iff X is σ-metacompact and countably metacompact.

Gulko has shown that for every Corson compact space K,

the space $C_p(K)$ is hereditarily σ-metacompact.

Dow, Junnila and Pelant (DJP) showed that

if $\bar{T}$ is the one-point compactification of a tree T,

then $C_p(\bar{T})$ is hereditarily σ-metacompact;

we also showed that $C_p(\beta\omega)$ is not σ-metacompact.

A space X is *countably metacompact*

if every countable open cover of X has a point-finite open refinement.

X is metacompact iff X is σ-metacompact and countably metacompact.

Through this connection, results on σ-metacompactness

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,
then $C_p(\bar{T})$ is hereditarily σ-metacompact;
we also showed that $C_p(\beta\omega)$ is not σ-metacompact.

A space X is *countably metacompact*
if every countable open cover of X has a point-finite open refinement.
X is metacompact iff X is σ-metacompact and countably metacompact.
Through this connection, results on σ-metacompactness
can sometimes be used to reduce problems on metacompactness

Gulko has shown that for every Corson compact space K,
the space $C_p(K)$ is hereditarily σ-metacompact.
Dow, Junnila and Pelant (DJP) showed that
if $\bar{T}$ is the one-point compactification of a tree T,
then $C_p(\bar{T})$ is hereditarily σ-metacompact;
we also showed that $C_p(\beta\omega)$ is not σ-metacompact.

A space X is *countably metacompact*
if every countable open cover of X has a point-finite open refinement.
X is metacompact iff X is σ-metacompact and countably metacompact.
Through this connection, results on σ-metacompactness
can sometimes be used to reduce problems on metacompactness
to problems on countable metacompactness.

-

Property $(*)$.

A family $\mathcal{L}$ of subsets of a space X is a k-*network* of X

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a *k-network* of X provided that for every compact closed subset C of X

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a k-*network* of X provided that for every compact closed subset C of X and for every nbhd G of the set C,

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a k-*network* of X provided that for every compact closed subset C of X and for every nbhd G of the set C, there exists a finite $\mathcal{K} \subset \mathcal{L}$

Property $(*)$.

A family $\mathcal{L}$ of subsets of a space X is a *k-network* of X provided that for every compact closed subset C of X and for every nbhd G of the set C, there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a k-*network* of X
provided that for every compact closed subset C of X
and for every nbhd G of the set C,
there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

In a locally compact space, k-networks are related to quasi-bases.

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a *k-network* of X
provided that for every compact closed subset C of X
and for every nbhd G of the set C,
there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

In a locally compact space, k-networks are related to quasi-bases.
A family $\mathcal{H}$ of subsets of a space X is a *quasi-base* of X

Property $(*)$.

A family $\mathcal{L}$ of subsets of a space X is a *k-network* of X
provided that for every compact closed subset C of X
and for every nbhd G of the set C,
there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

In a locally compact space, k-networks are related to quasi-bases.
A family $\mathcal{H}$ of subsets of a space X is a *quasi-base* of X
if $\{H \in \mathcal{H} : x \in \mathrm{Int} H\}$ is a nbhd base of x in X for each $x \in X$.

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a k-*network* of X
provided that for every compact closed subset C of X
and for every nbhd G of the set C,
there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

In a locally compact space, k-networks are related to quasi-bases.
A family $\mathcal{H}$ of subsets of a space X is a *quasi-base* of X
if $\{H \in \mathcal{H} : x \in \mathrm{Int} H\}$ is a nbhd base of x in X for each $x \in X$.
Every quasi-base is a k-network.

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a k-*network* of X
provided that for every compact closed subset C of X
and for every nbhd G of the set C,
there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

In a locally compact space, k-networks are related to quasi-bases.
A family $\mathcal{H}$ of subsets of a space X is a *quasi-base* of X
if $\{H \in \mathcal{H} : x \in \mathrm{Int} H\}$ is a nbhd base of x in X for each $x \in X$.
Every quasi-base is a k-network.

Lemma *A family $\mathcal{L}$ in a locally compact regular space X*

Property $(*)$**.**

A family $\mathcal{L}$ of subsets of a space X is a *k-network* of X
provided that for every compact closed subset C of X
and for every nbhd G of the set C,
there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

In a locally compact space, k-networks are related to quasi-bases.
A family $\mathcal{H}$ of subsets of a space X is a *quasi-base* of X
if $\{H \in \mathcal{H} : x \in \mathrm{Int} H\}$ is a nbhd base of x in X for each $x \in X$.
Every quasi-base is a k-network.

Lemma *A family $\mathcal{L}$ in a locally compact regular space X
is a k-network of X if, and only if,*

Property $(*)$.

A family $\mathcal{L}$ of subsets of a space X is a k-*network* of X
provided that for every compact closed subset C of X
and for every nbhd G of the set C,
there exists a finite $\mathcal{K} \subset \mathcal{L}$ such that $K \subset \bigcup \mathcal{K} \subset G$.

In a locally compact space, k-networks are related to quasi-bases.
A family $\mathcal{H}$ of subsets of a space X is a *quasi-base* of X
if $\{H \in \mathcal{H} : x \in \mathrm{Int} H\}$ is a nbhd base of x in X for each $x \in X$.
Every quasi-base is a k-network.

Lemma *A family $\mathcal{L}$ in a locally compact regular space X
is a k-network of X if, and only if,
the family $\left\{ \bigcup \mathcal{N} : \mathcal{N} \in [\mathcal{L}]^{<\omega} \right\}$ is a quasi-base of X.*

Corollary *Let $\mathcal{F}$ be a k-network of a locally compact regular X.*

Corollary *Let $\mathcal{F}$ be a k-network of a locally compact regular X. If $\mathcal{F}$ consists of closed sets, then for every $x \in X$,*

Corollary *Let* $\mathcal{F}$ *be a* k*-network of a locally compact regular* X. *If* $\mathcal{F}$ *consists of closed sets, then for every* $x \in X$, *the family* $\left\{\bigcup \mathcal{H} : \mathcal{H} \in \left[(\mathcal{F})_x\right]^{<\omega}\right\}$ *contains a nbhd base of* x.

Corollary *Let $\mathcal{F}$ be a k-network of a locally compact regular X. If $\mathcal{F}$ consists of closed sets, then for every $x \in X$, the family $\left\{\bigcup \mathcal{H} : \mathcal{H} \in \left[(\mathcal{F})_x\right]^{<\omega}\right\}$ contains a nbhd base of x.*

I will consider compact spaces which have a k-network $\mathcal{F}$ satisfying the following property.

Corollary *Let $\mathcal{F}$ be a k-network of a locally compact regular X. If $\mathcal{F}$ consists of closed sets, then for every $x \in X$, the family $\left\{\bigcup \mathcal{H} : \mathcal{H} \in \left[(\mathcal{F})_x\right]^{<\omega}\right\}$ contains a nbhd base of x.*

I will consider compact spaces which have a k-network $\mathcal{F}$ satisfying the following property.

(∗) *For all $x \in X$ and $S \in \mathcal{F}$, there exists a finite $E \subset S$ such that for each $F \in \mathcal{F}$, if $x \in F$ and $F \cap S \neq \emptyset$, then $F \cap E \neq \emptyset$.*

Corollary *Let $\mathcal{F}$ be a k-network of a locally compact regular X. If $\mathcal{F}$ consists of closed sets, then for every $x \in X$, the family $\left\{\bigcup \mathcal{H} : \mathcal{H} \in \left[(\mathcal{F})_x\right]^{<\omega}\right\}$ contains a nbhd base of x.*

I will consider compact spaces which have a k-network $\mathcal{F}$ satisfying the following property.

$(*)$ *For all $x \in X$ and $S \in \mathcal{F}$, there exists a finite $E \subset S$ such that for each $F \in \mathcal{F}$, if $x \in F$ and $F \cap S \neq \emptyset$, then $F \cap E \neq \emptyset$.*

The main result of this talk is the following

Corollary *Let* $\mathcal{F}$ *be a* k*-network of a locally compact regular* X. *If* $\mathcal{F}$ *consists of closed sets, then for every* $x \in X$, *the family* $\left\{\bigcup \mathcal{H} : \mathcal{H} \in \left[(\mathcal{F})_x\right]^{<\omega}\right\}$ *contains a nbhd base of* x.

I will consider compact spaces which have a k-network $\mathcal{F}$ satisfying the following property.

(∗) *For all* $x \in X$ *and* $S \in \mathcal{F}$, *there exists a finite* $E \subset S$ *such that for each* $F \in \mathcal{F}$, *if* $x \in F$ *and* $F \cap S \neq \emptyset$, *then* $F \cap E \neq \emptyset$.

The main result of this talk is the following

Theorem *If a compact space* K *has a* k*-network with property* $(*)$,

Corollary *Let $\mathcal{F}$ be a k-network of a locally compact regular X. If $\mathcal{F}$ consists of closed sets, then for every $x \in X$, the family $\left\{\bigcup \mathcal{H} : \mathcal{H} \in \left[(\mathcal{F})_x\right]^{<\omega}\right\}$ contains a nbhd base of x.*

I will consider compact spaces which have a k-network $\mathcal{F}$ satisfying the following property.

(∗) *For all $x \in X$ and $S \in \mathcal{F}$, there exists a finite $E \subset S$ such that for each $F \in \mathcal{F}$, if $x \in F$ and $F \cap S \neq \emptyset$, then $F \cap E \neq \emptyset$.*

The main result of this talk is the following

Theorem *If a compact space K has a k-network with property $(*)$, then $C_p(K)$ is hereditarily σ-metacompact.*

A standard proof shows that the property

A standard proof shows that the property of having a k-network with $(*)$

A standard proof shows that the property of having a k-network with $(*)$ is productive.

A standard proof shows that the property
of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$,

A standard proof shows that the property of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$, I look at conditions on families of sets which imply property $(*)$.

A standard proof shows that the property of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$, I look at conditions on families of sets which imply property $(*)$.

A family $\mathcal{N}$ of subsets of a space X is *pointwise closure-preserving*

A standard proof shows that the property of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$, I look at conditions on families of sets which imply property $(*)$.

A family $\mathcal{N}$ of subsets of a space X is *pointwise closure-preserving* provided that $(\mathcal{N})_x$ is closure-preserving for each $x \in X$.

A standard proof shows that the property

of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$,

I look at conditions on families of sets which imply property $(*)$.

A family $\mathcal{N}$ of subsets of a space X is *pointwise closure-preserving*

provided that $(\mathcal{N})_x$ is closure-preserving for each $x \in X$.

Lemma *If K is compact and $\mathcal{F}$ closure-preserving and closed in K,*

A standard proof shows that the property

of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$,

I look at conditions on families of sets which imply property $(*)$.

A family $\mathcal{N}$ of subsets of a space X is *pointwise closure-preserving*

provided that $(\mathcal{N})_x$ is closure-preserving for each $x \in X$.

Lemma *If K is compact and $\mathcal{F}$ closure-preserving and closed in K,*

then there exists a finite $E \subset K$

A standard proof shows that the property of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$, I look at conditions on families of sets which imply property $(*)$.

A family $\mathcal{N}$ of subsets of a space X is *pointwise closure-preserving* provided that $(\mathcal{N})_x$ is closure-preserving for each $x \in X$.

Lemma *If K is compact and $\mathcal{F}$ closure-preserving and closed in K, then there exists a finite $E \subset K$ such that $F \cap E \neq \emptyset$ for every $F \in \mathcal{F}$.*

A standard proof shows that the property
of having a k-network with $(*)$ is productive.

To find interesting classes of spaces which have k-networks with $(*)$,
I look at conditions on families of sets which imply property $(*)$.

A family $\mathcal{N}$ of subsets of a space X is *pointwise closure-preserving* provided that $(\mathcal{N})_x$ is closure-preserving for each $x \in X$.

Lemma *If K is compact and $\mathcal{F}$ closure-preserving and closed in K, then there exists a finite $E \subset K$ such that $F \cap E \neq \emptyset$ for every $F \in \mathcal{F}$.*

Corollary *Every pointwise closure-preserving family of compact closed sets has property $(*)$.*

-

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.
(In particular, if K has a pointwise closure-preserving closed quasi-base.)

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.
(In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point,

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.

(In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point, then X has a pointwise closure-preserving clopen base.

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.

(In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point, then X has a pointwise closure-preserving clopen base.

As a special case, for every cardinal κ,

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.

(In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point, then X has a pointwise closure-preserving clopen base.

As a special case, for every cardinal κ, the one-point compactification $A(\kappa)$ of the discrete space κ

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.
(In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point, then X has a pointwise closure-preserving clopen base.
As a special case, for every cardinal κ, the one-point compactification $A(\kappa)$ of the discrete space κ has a pointwise closure-preserving clopen base;

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.
(In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point,
then X has a pointwise closure-preserving clopen base.
As a special case, for every cardinal κ,
the one-point compactification $A(\kappa)$ of the discrete space κ
has a pointwise closure-preserving clopen base;
by previous observations, such a base has property $(*)$.

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network. (In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point, then X has a pointwise closure-preserving clopen base. As a special case, for every cardinal κ, the one-point compactification $A(\kappa)$ of the discrete space κ has a pointwise closure-preserving clopen base; by previous observations, such a base has property $(*)$.

It follows that for any two cardinals κ and λ,

It follows that $C_p(K)$ is hereditarily σ-metacompact if K is compact and K has a pointwise closure-preserving closed k-network.
(In particular, if K has a pointwise closure-preserving closed quasi-base.)

Note that if a T_1-space X has only one non-isolated point,
then X has a pointwise closure-preserving clopen base.
As a special case, for every cardinal κ,
the one-point compactification $A(\kappa)$ of the discrete space κ
has a pointwise closure-preserving clopen base;
by previous observations, such a base has property $(*)$.

It follows that for any two cardinals κ and λ,
the power $A(\kappa)^\lambda$ of $A(\kappa)$ has a clopen base with property $(*)$.

The following example shows that spaces of type $A(\kappa)^\lambda$

The following example shows that spaces of type $A(\kappa)^\lambda$ do not allways have a pointwise closure-preserving closed k-network.

The following example shows that spaces of type $A(\kappa)^\lambda$ do not allways have a pointwise closure-preserving closed k-network.

Example Consider the space $A(\omega_1)^\omega$.

The following example shows that spaces of type $A(\kappa)^\lambda$ do not allways have a pointwise closure-preserving closed k-network.

Example Consider the space $A(\omega_1)^\omega$.
Dow, Raminez Martinez and Tkachuk have shown that $\langle\infty,\infty,\ldots\rangle$ does not have a closure-preserving neighborhood base in $A(\omega_1)^\omega$.

The following example shows that spaces of type $A(\kappa)^\lambda$ do not allways have a pointwise closure-preserving closed k-network.

Example Consider the space $A(\omega_1)^\omega$.
Dow, Raminez Martinez and Tkachuk have shown that $\langle\infty,\infty,\ldots\rangle$ does not have a closure-preserving neighborhood base in $A(\omega_1)^\omega$.
It follows that $A(\omega_1)^\omega$ does not have a pointwise closure-preserving k-network consisting of closed sets. $\square$

Proposition *Every stratifiable space has a pointwise closure-preserving closed quasi-base.*

Proposition *Every stratifiable space has a pointwise closure-preserving closed quasi-base.*

Metrizable spaces are stratifiable

Proposition *Every stratifiable space has a pointwise closure-preserving closed quasi-base.*

Metrizable spaces are stratifiable
and compact stratifiable spaces are metrizable.

Proposition *Every stratifiable space has a pointwise closure-preserving closed quasi-base.*

Metrizable spaces are stratifiable
and compact stratifiable spaces are metrizable.

Corollary *A compact metrizable space has a closed k-network with property* $(*)$.

Proposition *Every stratifiable space has a pointwise closure-preserving closed quasi-base.*

Metrizable spaces are stratifiable
and compact stratifiable spaces are metrizable.

Corollary *A compact metrizable space has a closed k-network with property* $(*)$.

It will turn out that the above result can also be obtained
as a consequence of supercompactness of compact metrizable spaces.

Next I indicate some other properties of families of sets which imply property $(*)$.

Next I indicate some other properties of families of sets which imply property $(*)$.

Proposition *Each of the following conditions implies property* $(*)$ *for a family* $\mathcal{L}$ *of subsets of a space* X*:*

Next I indicate some other properties of families of sets which imply property $(*)$.

Proposition *Each of the following conditions implies property* $(*)$ *for a family* $\mathcal{L}$ *of subsets of a space* X*:*

(i) $\mathcal{L}$ *consists of connected rim-finite closed sets.*

Next I indicate some other properties of families of sets which imply property $(*)$.

Proposition *Each of the following conditions implies property* $(*)$ *for a family* $\mathcal{L}$ *of subsets of a space* X*:*

(i) $\mathcal{L}$ *consists of connected rim-finite closed sets.*

(ii) $\mathcal{L}$ *consists of closed* $\leq$*-intervals, where* $\leq$ *is a lattice ordering of* X*.*

Next I indicate some other properties of families of sets which imply property $(*)$.

Proposition *Each of the following conditions implies property* $(*)$ *for a family* $\mathcal{L}$ *of subsets of a space* X*:*

(i) $\mathcal{L}$ *consists of connected rim-finite closed sets.*

(ii) $\mathcal{L}$ *consists of closed* $\leq$*-intervals, where* $\leq$ *is a lattice ordering of* X*.*

(iii) $\mathcal{L}$ *is pointwise monotone*

Next I indicate some other properties of families of sets which imply property $(*)$.

Proposition *Each of the following conditions implies property* $(*)$ *for a family* $\mathcal{L}$ *of subsets of a space* X*:*

(i) $\mathcal{L}$ *consists of connected rim-finite closed sets.*

(ii) $\mathcal{L}$ *consists of closed* $\leq$*-intervals, where* $\leq$ *is a lattice ordering of* X*.*

(iii) $\mathcal{L}$ *is pointwise monotone (i.e., a rank-one family).*

Next I indicate some other properties of families of sets which imply property $(*)$.

Proposition *Each of the following conditions implies property* $(*)$ *for a family* $\mathcal{L}$ *of subsets of a space* X*:*

(i) $\mathcal{L}$ *consists of connected rim-finite closed sets.*

(ii) $\mathcal{L}$ *consists of closed* $\leq$*-intervals, where* $\leq$ *is a lattice ordering of* X*.*

(iii) $\mathcal{L}$ *is pointwise monotone (i.e., a rank-one family).*

Proof. If (i) holds, then L meets the finite subset ∂M of M whenever $L, M \in \mathcal{L}$, $L \cap M \neq \emptyset$ and $L \cap (X \setminus M) \neq \emptyset$.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$,

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

It follows,

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

It follows, since $c \leq x \leq d$,

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

It follows, since $c \leq x \leq d$, that $u \in [c, d]$.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

It follows, since $c \leq x \leq d$, that $u \in [c, d]$.

Under (iii), if $L, M \in \mathcal{L}$, $L \cap M \neq \emptyset$ and $L \cap (X \setminus M) \neq \emptyset$, then $M \subset L$. $\square$

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

It follows, since $c \leq x \leq d$, that $u \in [c, d]$.

Under (iii), if $L, M \in \mathcal{L}$, $L \cap M \neq \emptyset$ and $L \cap (X \setminus M) \neq \emptyset$, then $M \subset L$. □

Spaces with a k-network satisfying (i) above include “tree-like spaces”.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

It follows, since $c \leq x \leq d$, that $u \in [c, d]$.

Under (iii), if $L, M \in \mathcal{L}$, $L \cap M \neq \emptyset$ and $L \cap (X \setminus M) \neq \emptyset$, then $M \subset L$. $\square$

Spaces with a k-network satisfying (i) above include "tree-like spaces".

I will consider spaces with a k-network satisfying (ii) in more detail.

Assume that (ii) holds. Let $a, b, x \in X$ with $a \leq b$.

The point $u = (a \vee x) \wedge b$ belongs to the closed interval $[a, b]$.

Let $c, d \in X$ be such that $c \leq d$, $[c, d] \cap [a, b] \neq \emptyset$ and $x \in [c, d]$.

Since $[a, b] \cap [c, d] \neq \emptyset$, we have $a \leq d$ and $c \leq b$.

It follows, since $c \leq x \leq d$, that $u \in [c, d]$.

Under (iii), if $L, M \in \mathcal{L}$, $L \cap M \neq \emptyset$ and $L \cap (X \setminus M) \neq \emptyset$, then $M \subset L$. □

Spaces with a k-network satisfying (i) above include "tree-like spaces".

I will consider spaces with a k-network satisfying (ii) in more detail.

Spaces with a k-network satisfying (iii) include "non-Archimedean spaces", i.e., spaces with a pointwise monotone base.

Next I indicate some spaces which fail to have a k-network with $(*)$.

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.

By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.

By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.

It follows that $\beta\omega$ does not have a k-network with $(*)$.

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that
every point-finite open family is countable in $C_p(K_1)$.

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that every point-finite open family is countable in $C_p(K_1)$.
It follows that $C_p(K_1)$ is not hereditarily σ-metacompact:

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that every point-finite open family is countable in $C_p(K_1)$.
It follows that $C_p(K_1)$ is not hereditarily σ-metacompact:
otherwise $C_p(K_1)$ would be hereditarily Lindelöf

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that
every point-finite open family is countable in $C_p(K_1)$.
It follows that $C_p(K_1)$ is not hereditarily σ-metacompact:
otherwise $C_p(K_1)$ would be hereditarily Lindelöf
and it would follow from a result of Zenor

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that
every point-finite open family is countable in $C_p(K_1)$.
It follows that $C_p(K_1)$ is not hereditarily σ-metacompact:
otherwise $C_p(K_1)$ would be hereditarily Lindelöf
and it would follow from a result of Zenor
that K_1 would be hereditarily separable,

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that
every point-finite open family is countable in $C_p(K_1)$.
It follows that $C_p(K_1)$ is not hereditarily σ-metacompact:
otherwise $C_p(K_1)$ would be hereditarily Lindelöf
and it would follow from a result of Zenor
that K_1 would be hereditarily separable, which it is not.

Next I indicate some spaces which fail to have a k-network with $(*)$.

One example is provided by the separable compact space $\beta\omega$.
By a result of DJP, $C_p(\beta\omega)$ is not σ-metacompact.
It follows that $\beta\omega$ does not have a k-network with $(*)$.

There also exist scattered examples:
a scattered compact space K_1 was constructed by DJP such that every point-finite open family is countable in $C_p(K_1)$.
It follows that $C_p(K_1)$ is not hereditarily σ-metacompact:
otherwise $C_p(K_1)$ would be hereditarily Lindelöf
and it would follow from a result of Zenor
that K_1 would be hereditarily separable, which it is not.
Hence K_1 does not have a k-network with property $(*)$.

-

Above, the compact spaces failed to have a k-network with $(*)$

Above, the compact spaces failed to have a k-network with $(*)$ because their C_p-spaces were not hereditarily σ-metacompact.

Above, the compact spaces failed to have a k-network with $(*)$ because their C_p-spaces were not hereditarily σ-metacompact. Now I will describe another scattered example, Above, the compact spaces failed to have a k-network with $(*)$ because their C_p-spaces were not hereditarily σ-metacompact. Now I will describe another scattered example, whose C_p-space is hereditarily σ-metacompact.

Above, the compact spaces failed to have a k-network with $(*)$ because their C_p-spaces were not hereditarily σ-metacompact. Now I will describe another scattered example, whose C_p-space is hereditarily σ-metacompact.

Example Let T be a special Aronzajn-tree such that every chain of T has a supremum.

Above, the compact spaces failed to have a k-network with $(*)$
because their C_p-spaces were not hereditarily σ-metacompact.
Now I will describe another scattered example,
whose C_p-space is hereditarily σ-metacompact.

Example Let T be a special Aronzajn-tree
such that every chain of T has a supremum.
With the usual tree topology, T is a locally compact Hausdorff space.

Above, the compact spaces failed to have a k-network with $(*)$
because their C_p-spaces were not hereditarily σ-metacompact.
Now I will describe another scattered example,
whose C_p-space is hereditarily σ-metacompact.

Example Let T be a special Aronzajn-tree
such that every chain of T has a supremum.
With the usual tree topology, T is a locally compact Hausdorff space.

We can show that no Hausdorff compactification of T
has a k-network with $(*)$.

Above, the compact spaces failed to have a k-network with $(*)$
because their C_p-spaces were not hereditarily σ-metacompact.
Now I will describe another scattered example,
whose C_p-space is hereditarily σ-metacompact.

Example Let T be a special Aronzajn-tree
such that every chain of T has a supremum.
With the usual tree topology, T is a locally compact Hausdorff space.

We can show that no Hausdorff compactification of T
has a k-network with $(*)$.

In particular, the one-point compactification $\overline{T}$ of T
does not have a k-network with property $(*)$,

Above, the compact spaces failed to have a k-network with $(*)$
because their C_p-spaces were not hereditarily σ-metacompact.
Now I will describe another scattered example,
whose C_p-space is hereditarily σ-metacompact.

Example Let T be a special Aronzajn-tree
such that every chain of T has a supremum.
With the usual tree topology, T is a locally compact Hausdorff space.

We can show that no Hausdorff compactification of T
has a k-network with $(*)$.

In particular, the one-point compactification $\overline{T}$ of T
does not have a k-network with property $(*)$,
while a result of DJP shows that $C_p(\overline{T})$ is hereditarily σ-metacompact.

Supercompact spaces.

The results above can be applied to supercompact spaces.

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$, and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,
and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.
A space is *supercompact* if it has a binary subbase for closed sets.

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,
and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.
A space is *supercompact* if it has a binary subbase for closed sets.

Supercompact spaces form an important and extensive class of spaces.

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,
and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.
A space is *supercompact* if it has a binary subbase for closed sets.

Supercompact spaces form an important and extensive class of spaces.
For example, compact metrizable spaces,

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,
and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.
A space is *supercompact* if it has a binary subbase for closed sets.

Supercompact spaces form an important and extensive class of spaces.
For example, compact metrizable spaces, compact topological groups

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,
and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.
A space is *supercompact* if it has a binary subbase for closed sets.

Supercompact spaces form an important and extensive class of spaces. For example, compact metrizable spaces, compact topological groups and linearly orderable compact spaces are all supercompact.

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,
and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.
A space is *supercompact* if it has a binary subbase for closed sets.

Supercompact spaces form an important and extensive class of spaces.
For example, compact metrizable spaces, compact topological groups
and linearly orderable compact spaces are all supercompact.

I considered spaces defined in terms of special k-networks.

Supercompact spaces.

The results above can be applied to supercompact spaces.

A family $\mathcal{L}$ of sets is *linked* if $L \cap N \neq \emptyset$ for all $L, N \in \mathcal{L}$,
and $\mathcal{L}$ is *binary* if every linked subfamily of $\mathcal{L}$ is fixed.
A space is *supercompact* if it has a binary subbase for closed sets.

Supercompact spaces form an important and extensive class of spaces.
For example, compact metrizable spaces, compact topological groups
and linearly orderable compact spaces are all supercompact.

I considered spaces defined in terms of special k-networks.
The following simple and well-known results help us
to link supercompact spaces with those spaces.

Lemma *Let $\mathcal{F}$ be a family of closed subsets of a compact space X.*

Lemma *Let* $\mathcal{F}$ *be a family of closed subsets of a compact space* X.

A. If $\mathcal{F}$ *is a subbase of closed subsets of* X *and* $\mathcal{F}$ *is closed under finite intersections, then* $\mathcal{F}$ *is a* k*-network of* X.

Lemma *Let* $\mathcal{F}$ *be a family of closed subsets of a compact space* X.

A. If $\mathcal{F}$ *is a subbase of closed subsets of* X *and* $\mathcal{F}$ *is closed under finite intersections, then* $\mathcal{F}$ *is a* k*-network of* X.

B. If X *is* T_1 *and* $\mathcal{F}$ *is a* k*-network of* X, *then* $\mathcal{F}$ *is a subbase of closed subsets of* X.

Lemma *Let $\mathcal{F}$ be a family of closed subsets of a compact space X.*

A. If $\mathcal{F}$ is a subbase of closed subsets of X and $\mathcal{F}$ is closed under finite intersections, then $\mathcal{F}$ is a k-network of X.

B. If X is T_1 and $\mathcal{F}$ is a k-network of X, then $\mathcal{F}$ is a subbase of closed subsets of X.

For a binary family $\mathcal{F}$, also the family $\{\bigcap \mathcal{H} : \mathcal{H} \subset \mathcal{F}\}$ is binary.

Lemma *Let* $\mathcal{F}$ *be a family of closed subsets of a compact space* X *.*

A. If $\mathcal{F}$ *is a subbase of closed subsets of* X *and* $\mathcal{F}$ *is closed under finite intersections, then* $\mathcal{F}$ *is a* k*-network of* X *.*

B. If X *is* T_1 *and* $\mathcal{F}$ *is a* k*-network of* X *, then* $\mathcal{F}$ *is a subbase of closed subsets of* X *.*

For a binary family $\mathcal{F}$, also the family $\{\bigcap \mathcal{H} : \mathcal{H} \subset \mathcal{F}\}$ is binary. Hence the above lemma has the following consequence.

Lemma *Let* $\mathcal{F}$ *be a family of closed subsets of a compact space* X.

A. If $\mathcal{F}$ *is a subbase of closed subsets of* X *and* $\mathcal{F}$ *is closed under finite intersections, then* $\mathcal{F}$ *is a* k*-network of* X.

B. If X *is* T_1 *and* $\mathcal{F}$ *is a* k*-network of* X, *then* $\mathcal{F}$ *is a subbase of closed subsets of* X.

For a binary family $\mathcal{F}$, also the family $\{\bigcap \mathcal{H} : \mathcal{H} \subset \mathcal{F}\}$ is binary. Hence the above lemma has the following consequence.

Corollary *A. Every supercompact space has a binary closed* k*-network.*

Lemma *Let* $\mathcal{F}$ *be a family of closed subsets of a compact space* X *.*

A. If $\mathcal{F}$ *is a subbase of closed subsets of* X *and* $\mathcal{F}$ *is closed under finite intersections, then* $\mathcal{F}$ *is a* k*-network of* X *.*

B. If X *is* T_1 *and* $\mathcal{F}$ *is a* k*-network of* X *, then* $\mathcal{F}$ *is a subbase of closed subsets of* X *.*

For a binary family $\mathcal{F}$, also the family $\{\bigcap\mathcal{H} : \mathcal{H} \subset \mathcal{F}\}$ is binary. Hence the above lemma has the following consequence.

Corollary *A. Every supercompact space has a binary closed* k*-network.*

B. Every compact T_1*-space with a binary closed* k*-network is supercompact.*

-

Our next results show that the binary property, for a family of closed compact sets, can be characterized by the condition which we obtain by changing "finite set" to "singleton" in the definition of property $(*)$.

Our next results show that the binary property,

for a family of closed compact sets, can be characterized

by the condition which we obtain by changing "finite set"

to "singleton" in the definition of property $(*)$.

super $(*)$ *For all* $x \in X$ *and* $S \in \mathcal{F}$, *there exists* $e \in S$ *such that*

Our next results show that the binary property,
for a family of closed compact sets, can be characterized
by the condition which we obtain by changing “finite set”
to “singleton” in the definition of property $(*)$.

super $(*)$ *For all* $x \in X$ *and* $S \in \mathcal{F}$, *there exists* $e \in S$ *such that for each* $F \in \mathcal{F}$, *if* $x \in F$ *and* $F \cap S \neq \emptyset$, *then* $e \in F$.

Our next results show that the binary property, for a family of closed compact sets, can be characterized by the condition which we obtain by changing "finite set" to "singleton" in the definition of property $(*)$.

super $(*)$ *For all* $x \in X$ *and* $S \in \mathcal{F}$, *there exists* $e \in S$ *such that for each* $F \in \mathcal{F}$, *if* $x \in F$ *and* $F \cap S \neq \emptyset$, *then* $e \in F$.

A family $\mathcal{L}$ of sets is *centered* if every finite subfamily of $\mathcal{L}$ is fixed.

Our next results show that the binary property,

for a family of closed compact sets, can be characterized

by the condition which we obtain by changing "finite set"

to "singleton" in the definition of property $(*)$.

super $(*)$ *For all* $x \in X$ *and* $S \in \mathcal{F}$, *there exists* $e \in S$ *such that for each* $F \in \mathcal{F}$, *if* $x \in F$ *and* $F \cap S \neq \emptyset$, *then* $e \in F$.

A family $\mathcal{L}$ of sets is *centered* if every finite subfamily of $\mathcal{L}$ is fixed.

Lemma *Let* $\mathcal{A}$ *be a family of subsets of a set* S.
Every linked subfamily of $\mathcal{A}$ *is centered iff for all* $B \in \mathcal{A}$ *and* $s \in S$, *the family* $\{A \cap B : A \in (\mathcal{A})_s \text{ and } A \cap B \neq \emptyset\}$ *is centered.*

Proof. If every linked subfamily of $\mathcal{A}$ is centered, then $\mathcal{A}$ satisfies the condition stated in the proposition, because for all $s \in S$ and $B \in \mathcal{A}$, the family $\{B\} \cup \{A \in (\mathcal{A})_s : A \cap B \neq \emptyset\}$

Proof. If every linked subfamily of $\mathcal{A}$ is centered, then $\mathcal{A}$ satisfies the condition stated in the proposition, because for all $s \in S$ and $B \in \mathcal{A}$, the family $\{B\} \cup \{A \in (\mathcal{A})_s : A \cap B \neq \emptyset\}$ is linked and therefore centered.

Proof. If every linked subfamily of $\mathcal{A}$ is centered,
then $\mathcal{A}$ satisfies the condition stated in the proposition,
because for all $s \in S$ and $B \in \mathcal{A}$, the family
$\{B\} \cup \{A \in (\mathcal{A})_s : A \cap B \neq \emptyset\}$ is linked and therefore centered.

For the other direction,

Proof. If every linked subfamily of $\mathcal{A}$ is centered, then $\mathcal{A}$ satisfies the condition stated in the proposition, because for all $s \in S$ and $B \in \mathcal{A}$, the family $\{B\} \cup \{A \in (\mathcal{A})_s : A \cap B \neq \emptyset\}$ is linked and therefore centered.

For the other direction, assume that $\mathcal{A}$ satisfies the stated condition.

Proof. If every linked subfamily of $\mathcal{A}$ is centered, then $\mathcal{A}$ satisfies the condition stated in the proposition, because for all $s \in S$ and $B \in \mathcal{A}$, the family $\{B\} \cup \{A \in (\mathcal{A})_s : A \cap B \neq \emptyset\}$ is linked and therefore centered.

For the other direction, assume that $\mathcal{A}$ satisfies the stated condition. Let $\mathcal{H} \subset \mathcal{A}$ be linked. By induction on $n \in \mathbb{N}$, it is easy to show that each $\mathcal{J} \in [\mathcal{H}]^n$ is fixed. □

Since every centered family consisting of compact closed sets is fixed, the above lemma has the following consequence.

Since every centered family consisting of compact closed sets is fixed, the above lemma has the following consequence.

Proposition *A family $\mathcal{F}$ of compact closed subsets is binary*

Since every centered family consisting of compact closed sets is fixed, the above lemma has the following consequence.

Proposition *A family $\mathcal{F}$ of compact closed subsets is binary iff $\mathcal{F}$ has property super$(*)$.*

Since every centered family consisting of compact closed sets is fixed, the above lemma has the following consequence.

Proposition *A family $\mathcal{F}$ of compact closed subsets is binary iff $\mathcal{F}$ has property super$(*)$.*

Corollary *Each supercompact space has a closed k-network with $(*)$.*

Since every centered family consisting of compact closed sets is fixed, the above lemma has the following consequence.

Proposition *A family $\mathcal{F}$ of compact closed subsets is binary iff $\mathcal{F}$ has property super$(*)$.*

Corollary *Each supercompact space has a closed k-network with $(*)$.*

Corollary *For every supercompact space X, the space $C_p(X)$ is hereditarily σ-metacompact.*

Since every centered family consisting of compact closed sets is fixed, the above lemma has the following consequence.

Proposition *A family $\mathcal{F}$ of compact closed subsets is binary iff $\mathcal{F}$ has property super$(*)$.*

Corollary *Each supercompact space has a closed k-network with $(*)$.*

Corollary *For every supercompact space X, the space $C_p(X)$ is hereditarily σ-metacompact.*

The Aronzajn tree T considered earlier was such that no T_2-compactification of T has a closed k-network with $(*)$.

Since every centered family consisting of compact closed sets is fixed, the above lemma has the following consequence.

Proposition *A family $\mathcal{F}$ of compact closed subsets is binary iff $\mathcal{F}$ has property super$(*)$.*

Corollary *Each supercompact space has a closed k-network with $(*)$.*

Corollary *For every supercompact space X, the space $C_p(X)$ is hereditarily σ-metacompact.*

The Aronzajn tree T considered earlier was such that no T_2-compactification of T has a closed k-network with $(*)$. It follows that no T_2-compactification of T is supercompact.

-

The class of compact spaces which have a closed k-network with $(*)$ is strictly larger than the class of supercompact spaces.

The class of compact spaces which have a closed k-network with $(*)$ is strictly larger than the class of supercompact spaces.

Example Banakh, Kosztolowicz and Turek have shown that $\mathbb{I} \times A(\mathbf{c})$ contains a non-supercompact closed subspace X.

The class of compact spaces which have a closed k-network with $(*)$ is strictly larger than the class of supercompact spaces.

Example Banakh, Kosztolowicz and Turek have shown that $\mathbb{I} \times A(\mathbf{c})$ contains a non-supercompact closed subspace X.

The space $\mathbb{I}$ has a pointwise closure-preserving closed quasi-base

The class of compact spaces which have a closed k-network with $(*)$ is strictly larger than the class of supercompact spaces.

Example Banakh, Kosztolowicz and Turek have shown that $\mathbb{I} \times A(\mathbf{c})$ contains a non-supercompact closed subspace X.
The space $\mathbb{I}$ has a pointwise closure-preserving closed quasi-base and the space $A(\mathbf{c})$ has a pointwise closure-preserving clopen base.

The class of compact spaces which have a closed k-network with $(*)$ is strictly larger than the class of supercompact spaces.

Example Banakh, Kosztolowicz and Turek have shown that $\mathbb{I} \times A(\mathbf{c})$ contains a non-supercompact closed subspace X.
The space $\mathbb{I}$ has a pointwise closure-preserving closed quasi-base and the space $A(\mathbf{c})$ has a pointwise closure-preserving clopen base.
Hence $\mathbb{I} \times A(\mathbf{c})$ has a pointwise closure-preserving closed quasi-base,

The class of compact spaces which have a closed k-network with $(*)$ is strictly larger than the class of supercompact spaces.

Example Banakh, Kosztolowicz and Turek have shown that $\mathbb{I} \times A(\mathbf{c})$ contains a non-supercompact closed subspace X.
The space $\mathbb{I}$ has a pointwise closure-preserving closed quasi-base and the space $A(\mathbf{c})$ has a pointwise closure-preserving clopen base.
Hence $\mathbb{I} \times A(\mathbf{c})$ has a pointwise closure-preserving closed quasi-base, and so the closed subspace X has such a quasi-base $\mathcal{F}$.

The class of compact spaces which have a closed k-network with $(*)$ is strictly larger than the class of supercompact spaces.

Example Banakh, Kosztolowicz and Turek have shown that $\mathbb{I} \times A(\mathbf{c})$ contains a non-supercompact closed subspace X.
The space $\mathbb{I}$ has a pointwise closure-preserving closed quasi-base and the space $A(\mathbf{c})$ has a pointwise closure-preserving clopen base.
Hence $\mathbb{I} \times A(\mathbf{c})$ has a pointwise closure-preserving closed quasi-base, and so the closed subspace X has such a quasi-base $\mathcal{F}$.
The family $\mathcal{F}$ is a k-network of X and it has property $(*)$. $\square$

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.

For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.

For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

Proposition *A family $\mathcal{F}$ of non-empty compact closed subsets of X is binary*

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.

For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

Proposition *A family $\mathcal{F}$ of non-empty compact closed subsets of X is binary iff there exists a set Φ of retractions of X s.t. $\mathcal{R}(\Phi) = \mathcal{F}$*

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.
For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

Proposition *A family $\mathcal{F}$ of non-empty compact closed subsets of X is binary iff there exists a set Φ of retractions of X s.t. $\mathcal{R}(\Phi) = \mathcal{F}$ and $\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in \Phi$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.*

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.
For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

Proposition *A family $\mathcal{F}$ of non-empty compact closed subsets of X is binary iff there exists a set Φ of retractions of X s.t. $\mathcal{R}(\Phi) = \mathcal{F}$ and $\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in \Phi$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.*

Remark *The following hold for two retractions f and g of X:*

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.

For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

Proposition *A family $\mathcal{F}$ of non-empty compact closed subsets of X is binary iff there exists a set Φ of retractions of X s.t. $\mathcal{R}(\Phi) = \mathcal{F}$ and $\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in \Phi$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.*

Remark *The following hold for two retractions f and g of X:*

(a) if $f \circ g \circ f = g \circ f$, then $f(X) \cap g(X) \neq \emptyset$;

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.

For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

Proposition *A family $\mathcal{F}$ of non-empty compact closed subsets of X is binary iff there exists a set Φ of retractions of X s.t. $\mathcal{R}(\Phi) = \mathcal{F}$ and $\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in \Phi$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.*

Remark *The following hold for two retractions f and g of X:*

(a) if $f \circ g \circ f = g \circ f$, then $f(X) \cap g(X) \neq \emptyset$;

(b) if $f \circ g \circ f = g \circ f$, then the mapping $g \circ f$ is a retraction.

The above characterization of binary families of compact closed sets can be used to present such families in terms of retractions.
For a set Γ of mappings on X, let $\mathcal{R}(\Gamma) = \{g(X) : g \in \Gamma\}$.

Proposition *A family $\mathcal{F}$ of non-empty compact closed subsets of X is binary iff there exists a set Φ of retractions of X s.t. $\mathcal{R}(\Phi) = \mathcal{F}$ and $\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in \Phi$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.*

Remark *The following hold for two retractions f and g of X:*

(a) if $f \circ g \circ f = g \circ f$, then $f(X) \cap g(X) \neq \emptyset$;

(b) if $f \circ g \circ f = g \circ f$, then the mapping $g \circ f$ is a retraction.

(c) if f and g commute, then $f \circ g \circ f = g \circ f$.

-

So every supercompact space X has a closed k-network $\{\varphi(X) : \varphi \in \Phi\}$,

So every supercompact space X has a closed k-network $\{\varphi(X) : \varphi \in \Phi\}$, where Φ is a set of retractions of X such that

$\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in F$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.

So every supercompact space X has a closed k-network $\{\varphi(X) : \varphi \in \Phi\}$, where Φ is a set of retractions of X such that

$\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in F$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.

The construction given for Φ does not produce continuous mappings

So every supercompact space X has a closed k-network $\{\varphi(X) : \varphi \in \Phi\}$, where Φ is a set of retractions of X such that
$\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in F$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.
The construction given for Φ does not produce continuous mappings and I do not know if each supercompact space X has such a set Φ

So every supercompact space X has a closed k-network $\{\varphi(X) : \varphi \in \Phi\}$, where Φ is a set of retractions of X such that
$\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in F$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.
The construction given for Φ does not produce continuous mappings and I do not know if each supercompact space X has such a set Φ consisting of *continuous* retractions.

So every supercompact space X has a closed k-network $\{\varphi(X) : \varphi \in \Phi\}$, where Φ is a set of retractions of X such that

$\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in F$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.

The construction given for Φ does not produce continuous mappings and I do not know if each supercompact space X has such a set Φ consisting of *continuous* retractions.

However, I can indicate one important class of supercompact spaces,

So every supercompact space X has a closed k-network $\{\varphi(X) : \varphi \in \Phi\}$, where Φ is a set of retractions of X such that

$\varphi \circ \psi \circ \varphi = \psi \circ \varphi$ whenever $\varphi, \psi \in F$ and $\varphi(X) \cap \psi(X) \neq \emptyset$.

The construction given for Φ does not produce continuous mappings and I do not know if each supercompact space X has such a set Φ consisting of *continuous* retractions.

However, I can indicate one important class of supercompact spaces, which have k-networks defined in this way
by a set of continuous retractions.

Example Let $(X, \leq)$ be a lattice,

Example Let $(X, \leq)$ be a lattice,

and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.

Example Let $(X, \leq)$ be a lattice,

and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.

The proof of an earlier proposition shows that $\mathcal{F}$ has property $super(*)$.

Example Let $(X, \leq)$ be a lattice,

and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.

The proof of an earlier proposition shows that $\mathcal{F}$ has property $super(*)$.

The family $\mathcal{F}$ can also be defined in terms of a set Φ

of continuous retractions $X \to X$ satisfying the previous condition.

Example Let $(X, \leq)$ be a lattice,

and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.
The proof of an earlier proposition shows that $\mathcal{F}$ has property $super(*)$.

The family $\mathcal{F}$ can also be defined in terms of a set Φ
of continuous retractions $X \to X$ satisfying the previous condition.
For all $a, b \in X$ with $a \leq b$, define $f_{a,b} : X \to X$ by the condition
$f_{a,b}(x) = (a \vee x) \wedge b$.

Example Let $(X, \leq)$ be a lattice,

and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.

The proof of an earlier proposition shows that $\mathcal{F}$ has property *super*$(*)$.

The family $\mathcal{F}$ can also be defined in terms of a set Φ

of continuous retractions $X \to X$ satisfying the previous condition.

For all $a, b \in X$ with $a \leq b$, define $f_{a,b} : X \to X$ by the condition

$f_{a,b}(x) = (a \vee x) \wedge b$.

It is easy to see that $f_{a,b}$ is a retraction and $f_{a,b}(X) = [a, b]$.

Example Let $(X, \leq)$ be a lattice,
and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.
The proof of an earlier proposition shows that $\mathcal{F}$ has property *super*$(*)$.

The family $\mathcal{F}$ can also be defined in terms of a set Φ
of continuous retractions $X \to X$ satisfying the previous condition.
For all $a, b \in X$ with $a \leq b$, define $f_{a,b} : X \to X$ by the condition
$f_{a,b}(x) = (a \vee x) \wedge b$.
It is easy to see that $f_{a,b}$ is a retraction and $f_{a,b}(X) = [a, b]$.
Let $\Phi = \{f_{a,b} : a, b \in X \text{ and } a \leq b\}$.

Example Let $(X, \leq)$ be a lattice,

and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.

The proof of an earlier proposition shows that $\mathcal{F}$ has property *super*$(*)$.

The family $\mathcal{F}$ can also be defined in terms of a set Φ

of continuous retractions $X \to X$ satisfying the previous condition.

For all $a, b \in X$ with $a \leq b$, define $f_{a,b} : X \to X$ by the condition

$f_{a,b}(x) = (a \vee x) \wedge b$.

It is easy to see that $f_{a,b}$ is a retraction and $f_{a,b}(X) = [a, b]$.

Let $\Phi = \{f_{a,b} : a, b \in X \text{ and } a \leq b\}$.

To show that Φ satisfies the previous condition,

Example Let $(X, \leq)$ be a lattice,
and let $\mathcal{F}$ be the family consisting of all closed intervals of $(X, \leq)$.
The proof of an earlier proposition shows that $\mathcal{F}$ has property $super(*)$.

The family $\mathcal{F}$ can also be defined in terms of a set Φ
of continuous retractions $X \to X$ satisfying the previous condition.
For all $a, b \in X$ with $a \leq b$, define $f_{a,b} : X \to X$ by the condition
$f_{a,b}(x) = (a \vee x) \wedge b$.
It is easy to see that $f_{a,b}$ is a retraction and $f_{a,b}(X) = [a, b]$.
Let $\Phi = \{f_{a,b} : a, b \in X \text{ and } a \leq b\}$.
To show that Φ satisfies the previous condition,
let $f_{a,b}, f_{c,d} \in \Phi$ be such that $f_{a,b}(X) \cap f_{c,d}(X) \neq \emptyset$.

-

Then $[a,b] \cap [c,d] \neq \emptyset$, and an earlier proof shows that for every $y \in [c,d]$, we have $f_{a,b}(y) \in [c,d]$.

Then $[a,b] \cap [c,d] \neq \emptyset$, and an earlier proof shows that

for every $y \in [c,d]$, we have $f_{a,b}(y) \in [c,d]$.

For every $x \in X$, we have $f_{c,d}(x) \in [c,d]$ and so $f_{a,b}(f_{c,d}(x)) \in [c,d]$.

Then $[a,b] \cap [c,d] \neq \emptyset$, and an earlier proof shows that for every $y \in [c,d]$, we have $f_{a,b}(y) \in [c,d]$.

For every $x \in X$, we have $f_{c,d}(x) \in [c,d]$ and so $f_{a,b}(f_{c,d}(x)) \in [c,d]$.

Since $f_{c,d}$ is a retraction and $f_{c,d}(X) = [c,d]$, it follows that $f_{c,d}(f_{a,b}(f_{c,d}(x))) = f_{a,b}(f_{c,d}(x))$ for every $x \in X$,

Then $[a,b] \cap [c,d] \neq \emptyset$, and an earlier proof shows that for every $y \in [c,d]$, we have $f_{a,b}(y) \in [c,d]$.
For every $x \in X$, we have $f_{c,d}(x) \in [c,d]$ and so $f_{a,b}(f_{c,d}(x)) \in [c,d]$.
Since $f_{c,d}$ is a retraction and $f_{c,d}(X) = [c,d]$, it follows that $f_{c,d}(f_{a,b}(f_{c,d}(x))) = f_{a,b}(f_{c,d}(x))$ for every $x \in X$, i.e., that $f_{c,d} \circ f_{a,b} \circ f_{c,d} = f_{a,b} \circ f_{c,d}$.

Then $[a, b] \cap [c, d] \neq \emptyset$, and an earlier proof shows that
for every $y \in [c, d]$, we have $f_{a,b}(y) \in [c, d]$.
For every $x \in X$, we have $f_{c,d}(x) \in [c, d]$ and so $f_{a,b}(f_{c,d}(x)) \in [c, d]$.
Since $f_{c,d}$ is a retraction and $f_{c,d}(X) = [c, d]$, it follows
that $f_{c,d}(f_{a,b}(f_{c,d}(x))) = f_{a,b}(f_{c,d}(x))$ for every $x \in X$,
i.e., that $f_{c,d} \circ f_{a,b} \circ f_{c,d} = f_{a,b} \circ f_{c,d}$.

The topology of X which has the family $\mathcal{F}$ as a subbase of closed sets

Then $[a, b] \cap [c, d] \neq \emptyset$, and an earlier proof shows that for every $y \in [c, d]$, we have $f_{a,b}(y) \in [c, d]$.
For every $x \in X$, we have $f_{c,d}(x) \in [c, d]$ and so $f_{a,b}(f_{c,d}(x)) \in [c, d]$.
Since $f_{c,d}$ is a retraction and $f_{c,d}(X) = [c, d]$, it follows that $f_{c,d}(f_{a,b}(f_{c,d}(x))) = f_{a,b}(f_{c,d}(x))$ for every $x \in X$, i.e., that $f_{c,d} \circ f_{a,b} \circ f_{c,d} = f_{a,b} \circ f_{c,d}$.

The topology of X which has the family $\mathcal{F}$ as a subbase of closed sets is called the *interval topology* of the lattice $(X, \leq)$.

Then $[a, b] \cap [c, d] \neq \emptyset$, and an earlier proof shows that for every $y \in [c, d]$, we have $f_{a,b}(y) \in [c, d]$.
For every $x \in X$, we have $f_{c,d}(x) \in [c, d]$ and so $f_{a,b}(f_{c,d}(x)) \in [c, d]$.
Since $f_{c,d}$ is a retraction and $f_{c,d}(X) = [c, d]$, it follows that $f_{c,d}(f_{a,b}(f_{c,d}(x))) = f_{a,b}(f_{c,d}(x))$ for every $x \in X$, i.e., that $f_{c,d} \circ f_{a,b} \circ f_{c,d} = f_{a,b} \circ f_{c,d}$.

The topology of X which has the family $\mathcal{F}$ as a subbase of closed sets is called the *interval topology* of the lattice $(X, \leq)$.
This topology was defined by Frink.

Then $[a,b] \cap [c,d] \neq \emptyset$, and an earlier proof shows that
for every $y \in [c,d]$, we have $f_{a,b}(y) \in [c,d]$.
For every $x \in X$, we have $f_{c,d}(x) \in [c,d]$ and so $f_{a,b}(f_{c,d}(x)) \in [c,d]$.
Since $f_{c,d}$ is a retraction and $f_{c,d}(X) = [c,d]$, it follows
that $f_{c,d}(f_{a,b}(f_{c,d}(x))) = f_{a,b}(f_{c,d}(x))$ for every $x \in X$,
i.e., that $f_{c,d} \circ f_{a,b} \circ f_{c,d} = f_{a,b} \circ f_{c,d}$.

The topology of X which has the family $\mathcal{F}$ as a subbase of closed sets is called the *interval topology* of the lattice $(X, \leq)$.
This topology was defined by Frink.
The importance of the interval topology is due to the result of Frink that the interval topology of a complete lattice is compact.

-

When the interval topology is compact, it follows from the foregoing

When the interval topology is compact, it follows from the foregoing that the defining subbase of closed sets is binary.

When the interval topology is compact, it follows from the foregoing that the defining subbase of closed sets is binary.
Hence we have the result of van Mill and Schrijver
that a complete lattice with interval topology is supercompact.

When the interval topology is compact, it follows from the foregoing
that the defining subbase of closed sets is binary.
Hence we have the result of van Mill and Schrijver
that a complete lattice with interval topology is supercompact.
Note that if the interval topology is compact,

When the interval topology is compact, it follows from the foregoing that the defining subbase of closed sets is binary.
Hence we have the result of van Mill and Schrijver that a complete lattice with interval topology is supercompact.
Note that if the interval topology is compact, then the defining closed subbase is also a k-network.

When the interval topology is compact, it follows from the foregoing that the defining subbase of closed sets is binary.
Hence we have the result of van Mill and Schrijver that a complete lattice with interval topology is supercompact.
Note that if the interval topology is compact, then the defining closed subbase is also a k-network.

It was observed above that the equation $f \circ g \circ f = g \circ f$ holds if the retractions f and g commute.

When the interval topology is compact, it follows from the foregoing that the defining subbase of closed sets is binary.
Hence we have the result of van Mill and Schrijver that a complete lattice with interval topology is supercompact.
Note that if the interval topology is compact, then the defining closed subbase is also a k-network.

It was observed above that the equation $f \circ g \circ f = g \circ f$ holds if the retractions f and g commute.
The next result shows that, for a large class of lattices, the defining subbase of the interval topology can be given in terms of a "conditionally commuting" family of retractions.

Proposition *The following conditions are mutually equivalent when* $(X, \leq)$ *is a lattice and the retractions* $f_{a,b}$ *are as above.*

Proposition *The following conditions are mutually equivalent when* $(X, \leq)$ *is a lattice and the retractions* $f_{a,b}$ *are as above.*

A. The lattice $(X, \leq)$ *is modular.*

Proposition *The following conditions are mutually equivalent when* $(X, \leq)$ *is a lattice and the retractions* $f_{a,b}$ *are as above.*

A. *The lattice* $(X, \leq)$ *is modular.*

B. $f_{a,b} \circ f_{c,d} = f_{a \vee c, b \wedge d}$ *whenever* $f_{a,b}(X) \cap f_{c,d}(X) \neq \emptyset$.

Proposition *The following conditions are mutually equivalent when* $(X, \leq)$ *is a lattice and the retractions* $f_{a,b}$ *are as above.*

A. The lattice $(X, \leq)$ *is modular.*

B. $f_{a,b} \circ f_{c,d} = f_{a \vee c, b \wedge d}$ *whenever* $f_{a,b}(X) \cap f_{c,d}(X) \neq \emptyset$.

C. $f_{a,b} \circ f_{c,d} = f_{c,d} \circ f_{a,b}$ *whenever* $f_{a,b}(X) \cap f_{c,d}(X) \neq \emptyset$.

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable*

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable* if $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is point–finitely expandable

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable* if $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is point–finitely expandable (i.e., for each n, there exist open sets $G_L \supset L$, for $L \in \mathcal{L}_n$

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable* if $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is point–finitely expandable (i.e., for each n, there exist open sets $G_L \supset L$, for $L \in \mathcal{L}_n$ s.t. for every $x \in X$, the family $\{L \in \mathcal{L}_n : x \in G_L\}$ is finite).

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable* if $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is point–finitely expandable (i.e., for each n, there exist open sets $G_L \supset L$, for $L \in \mathcal{L}_n$ s.t. for every $x \in X$, the family $\{L \in \mathcal{L}_n : x \in G_L\}$ is finite).

Proposition *(DJP) TFAE for a compact space K:*

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable* if $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is point–finitely expandable (i.e., for each n, there exist open sets $G_L \supset L$, for $L \in \mathcal{L}_n$ s.t. for every $x \in X$, the family $\{L \in \mathcal{L}_n : x \in G_L\}$ is finite).

Proposition *(DJP) TFAE for a compact space K:*

A. $C_p(K)$ has a σ-point-finitely expandable network.

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable* if $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is point–finitely expandable (i.e., for each n, there exist open sets $G_L \supset L$, for $L \in \mathcal{L}_n$ s.t. for every $x \in X$, the family $\{L \in \mathcal{L}_n : x \in G_L\}$ is finite).

Proposition *(DJP) TFAE for a compact space K:*

A. $C_p(K)$ has a σ-point-finitely expandable network.

B. $C_p(K)$ is hereditarily σ-metacompact and norm-σ-fragmented.

Function spaces with a σ-point–finitely expandable network.

At the end, I will briefly consider a strong sufficient condition for hereditary σ-metacompactness.

A family $\mathcal{L}$ of subsets of a space X is *σ–point–finitely expandable* if $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$, where each $\mathcal{L}_n$ is point–finitely expandable (i.e., for each n, there exist open sets $G_L \supset L$, for $L \in \mathcal{L}_n$ s.t. for every $x \in X$, the family $\{L \in \mathcal{L}_n : x \in G_L\}$ is finite).

Proposition *(DJP) TFAE for a compact space K:*

A. $C_p(K)$ has a σ-point-finitely expandable network.

B. $C_p(K)$ is hereditarily σ-metacompact and norm-σ-fragmented.

C. $C(K)$ has a network which is σ-point-finitely expandable in $C_p(K)$.

Some compact spaces K such that $C_p(K)$ has a σ-point-finitely expandable network were indicated by DJP; for example, all Corson compact spaces K have this property.

Some compact spaces K such that $C_p(K)$ has a
σ-point-finitely expandable network were indicated by DJP;
for example, all Corson compact spaces K have this property.
I will now indicate further examples of such compact spaces.

Some compact spaces K such that $C_p(K)$ has a
σ-point-finitely expandable network were indicated by DJP;
for example, all Corson compact spaces K have this property.
I will now indicate further examples of such compact spaces.

Proposition *If the compactum K has a closed k-network with $(*)$, then $C_p(K)$ has a σ-point-finitely expandable network iff $C_p(K)$ is norm-σ-fragmented.*

Some compact spaces K such that $C_p(K)$ has a σ-point-finitely expandable network were indicated by DJP; for example, all Corson compact spaces K have this property. I will now indicate further examples of such compact spaces.

Proposition *If the compactum K has a closed k-network with $(*)$, then $C_p(K)$ has a σ-point-finitely expandable network iff $C_p(K)$ is norm-σ-fragmented.*

Corollary *For every $i \in I$, let K_i be a compact space s.t. K_i has a closed k-network with property $(*)$ and $C_p(K_i)$ has a σ-point-finitely expandable network. Then $C_p(\prod_{i\in I} K_i)$ has such a network.*

Some compact spaces K such that $C_p(K)$ has a σ-point-finitely expandable network were indicated by DJP; for example, all Corson compact spaces K have this property. I will now indicate further examples of such compact spaces.

Proposition *If the compactum K has a closed k-network with $(*)$, then $C_p(K)$ has a σ-point-finitely expandable network iff $C_p(K)$ is norm-σ-fragmented.*

Corollary *For every $i \in I$, let K_i be a compact space s.t. K_i has a closed k-network with property $(*)$ and $C_p(K_i)$ has a σ-point-finitely expandable network. Then $C_p(\prod_{i\in I} K_i)$ has such a network.*

Proof. Use the above results and a theorem of Kenderov and Moors which shows that $C_p(\prod_{i\in I} K_i)$ is norm-σ-fragmented. $\square$

This result can be applied to dyadic and polyadic spaces. Recall that a topological space X is *dyadic* provided that X is the continuous image of $\{0,1\}^\kappa$ for some cardinal κ.

This result can be applied to dyadic and polyadic spaces.

Recall that a topological space X is *dyadic* provided that X is the continuous image of $\{0,1\}^\kappa$ for some cardinal κ.

The more general class of polyadic spaces was defined by Mrowka: X is *polyadic* if X is the continuous image of $A(\kappa)^\lambda$ for some cardinals κ and λ.

This result can be applied to dyadic and polyadic spaces.

Recall that a topological space X is *dyadic* provided that X is the continuous image of $\{0,1\}^\kappa$ for some cardinal κ.

The more general class of polyadic spaces was defined by Mrowka: X is *polyadic* if X is the continuous image of $A(\kappa)^\lambda$ for some cardinals κ and λ.

Theorem *Let X be a polyadic space. Then $C_p(X)$ has a σ-point-finitely expandable network.*

By the famous Kuzminov - Ivanovski Theorem,

every compact Hausdorff topological group is dyadic.

As a consequence, the following resul obtains:

By the famous Kuzminov - Ivanovski Theorem,

every compact Hausdorff topological group is dyadic.

As a consequence, the following resul obtains:

Corollary *$C_p(G)$ has a σ-point-finitely expandable network for every compact Hausdorff topological group G.*

By the famous Kuzminov - Ivanovski Theorem,
every compact Hausdorff topological group is dyadic.
As a consequence, the following resul obtains:

Corollary *$C_p(G)$ has a σ-point-finitely expandable network for every compact Hausdorff topological group G.*

Mills has shown that
all compact Hausdorff topological groups are supercompact.

By the famous Kuzminov - Ivanovski Theorem,
every compact Hausdorff topological group is dyadic.
As a consequence, the following resul obtains:

Corollary $C_p(G)$ *has a* σ*-point-finitely expandable network for every compact Hausdorff topological group* G.

Mills has shown that
all compact Hausdorff topological groups are supercompact.
Bell has shown that there are dyadic spaces which are not supercompact.

By the famous Kuzminov - Ivanovski Theorem,
every compact Hausdorff topological group is dyadic.
As a consequence, the following resul obtains:

Corollary *$C_p(G)$ has a σ-point-finitely expandable network for every compact Hausdorff topological group G.*

Mills has shown that
all compact Hausdorff topological groups are supercompact.
Bell has shown that there are dyadic spaces which are not supercompact.

Some results about supercompact spaces K with special properties imply that $C_p(K)$ is norm-σ-fragmented.

By the famous Kuzminov - Ivanovski Theorem,
every compact Hausdorff topological group is dyadic.
As a consequence, the following resul obtains:

Corollary *$C_p(G)$ has a σ-point-finitely expandable network for every compact Hausdorff topological group G.*

Mills has shown that
all compact Hausdorff topological groups are supercompact.
Bell has shown that there are dyadic spaces which are not supercompact.

Some results about supercompact spaces K with special properties imply that $C_p(K)$ is norm-σ-fragmented.
We can apply the previous results on such spaces to exhibit further $C_p(K)$-spaces with a σ-point-finitely expandable network.

-

The basic example of a supercompact space is a compact orderable space.

The basic example of a supercompact space is a compact orderable space.

A result of Jayne, Namioka and Rogers shows that

$C_p(X)$ is norm-σ-fragmentable for a compact orderable space X.

The basic example of a supercompact space is a compact orderable space.

A result of Jayne, Namioka and Rogers shows that

$C_p(X)$ is norm-σ-fragmentable for a compact orderable space X.

It follows that $C_p(X)$ has a σ-point-finitely expandable network.

The basic example of a supercompact space is a compact orderable space.
A result of Jayne, Namioka and Rogers shows that
$C_p(X)$ is norm-σ-fragmentable for a compact orderable space X.
It follows that $C_p(X)$ has a σ-point-finitely expandable network.
If Y is the continuous image of X, then $C_p(Y)$ embeds in $C_p(X)$
and therefore also $C_p(Y)$ has such a network.

The basic example of a supercompact space is a compact orderable space.
A result of Jayne, Namioka and Rogers shows that
$C_p(X)$ is norm-σ-fragmentable for a compact orderable space X.
It follows that $C_p(X)$ has a σ-point-finitely expandable network.
If Y is the continuous image of X, then $C_p(Y)$ embeds in $C_p(X)$
and therefore also $C_p(Y)$ has such a network.

A deep result of M.E. Rudin states that a compact T_2-space Z
is the continuous image of an orderable compact space
iff Z is monotonically normal.

The basic example of a supercompact space is a compact orderable space.

A result of Jayne, Namioka and Rogers shows that
$C_p(X)$ is norm-σ-fragmentable for a compact orderable space X.
It follows that $C_p(X)$ has a σ-point-finitely expandable network.
If Y is the continuous image of X, then $C_p(Y)$ embeds in $C_p(X)$ and therefore also $C_p(Y)$ has such a network.

A deep result of M.E. Rudin states that a compact T_2-space Z is the continuous image of an orderable compact space iff Z is monotonically normal.
A remarkable result of Bula, Nikiel, Tuncali and Tymchatum shows that continuous images of compact orderable spaces are supercompact.

The basic example of a supercompact space is a compact orderable space.
A result of Jayne, Namioka and Rogers shows that
$C_p(X)$ is norm-σ-fragmentable for a compact orderable space X.
It follows that $C_p(X)$ has a σ-point-finitely expandable network.
If Y is the continuous image of X, then $C_p(Y)$ embeds in $C_p(X)$
and therefore also $C_p(Y)$ has such a network.

A deep result of M.E. Rudin states that a compact T_2-space Z
is the continuous image of an orderable compact space
iff Z is monotonically normal.
A remarkable result of Bula, Nikiel, Tuncali and Tymchatum shows that
continuous images of compact orderable spaces are supercompact.

A combination of these results and the previous considerations
gives the the following result.

-

Proposition *If each* X_i *is a monotonically normal compactum, then* $C_p\left(\prod_{i\in I} X_i\right)$ *has a* σ*-point-finitely expandable network.*

Proposition *If each* X_i *is a monotonically normal compactum, then* $C_p\left(\prod_{i\in I} X_i\right)$ *has a* σ*-point-finitely expandable network.*

Theorem *Let* K *be a closed sublattice of a product of finitely many linearly ordered compact spaces. Then* $C_p(K)$ *has a* σ*-point-finitely expandable network.*

Proposition *If each X_i is a monotonically normal compactum, then $C_p\left(\prod_{i\in I} X_i\right)$ has a σ-point-finitely expandable network.*

Theorem *Let K be a closed sublattice of a product of finitely many linearly ordered compact spaces. Then $C_p(K)$ has a σ-point-finitely expandable network.*

Proof. The relative topology of the lattice K in the product
is the interval topology, and hence K is supercompact.

Proposition *If each* X_i *is a monotonically normal compactum, then* $C_p\left(\prod_{i\in I} X_i\right)$ *has a* σ*-point-finitely expandable network.*

Theorem *Let* K *be a closed sublattice of a product of finitely many linearly ordered compact spaces. Then* $C_p(K)$ *has a* σ*-point-finitely expandable network.*

Proof. The relative topology of the lattice K in the product
is the interval topology, and hence K is supercompact.
A result of Kubis, Molto and Troyanski shows
that $C_p(K)$ is norm-σ-fragmented. $\square$

A final observation.

Example The Helly space H is the subspace of $\mathbb{I}^{\mathbb{I}}$ consisting of all non-decreasing functions.

A final observation.

Example The Helly space H is the subspace of $\mathbb{I}^{\mathbb{I}}$ consisting of all non-decreasing functions.
Clearly, H is a closed sublattice of $\mathbb{I}^{\mathbb{I}}$,
and results of Frink show that the topology
which H inherits from $\mathbb{I}^{\mathbb{I}}$ coincides with
the interval topology of the lattice H.

A final observation.

Example The Helly space H is the subspace of $\mathbb{I}^{\mathbb{I}}$ consisting of all non-decreasing functions.
Clearly, H is a closed sublattice of $\mathbb{I}^{\mathbb{I}}$,
and results of Frink show that the topology which H inherits from $\mathbb{I}^{\mathbb{I}}$ coincides with the interval topology of the lattice H.
It follows that H is supercompact.

A final observation.

Example The Helly space H is the subspace of $\mathbb{I}^{\mathbb{I}}$ consisting of all non-decreasing functions.
Clearly, H is a closed sublattice of $\mathbb{I}^{\mathbb{I}}$,
and results of Frink show that the topology
which H inherits from $\mathbb{I}^{\mathbb{I}}$ coincides with
the interval topology of the lattice H.
It follows that H is supercompact.
Kortezov has shown in that $C_p(H)$ is norm-σ-fragmented.

A final observation.

Example The Helly space H is the subspace of $\mathbb{I}^{\mathbb{I}}$ consisting of all non-decreasing functions.
Clearly, H is a closed sublattice of $\mathbb{I}^{\mathbb{I}}$, and results of Frink show that the topology which H inherits from $\mathbb{I}^{\mathbb{I}}$ coincides with the interval topology of the lattice H.
It follows that H is supercompact.
Kortezov has shown in that $C_p(H)$ is norm-σ-fragmented.
Hence $C_p(H)$ has a σ-point-finitely expandable network. $\square$