Weak diamonds and topology

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Typical problem to consider...

There are two powerful paradigms in set theory for solving (e.g. topological) problems

- The Axiom of constructibility V=L, and
- forcing axioms (MA, PFA, MM,...)

We shall look at problems not settled by these (or rather "settled in the same way"). Usually problems of the form:

Is there a topological space (or a family of spaces, or a combinatorial object) with property P?

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Typical analysis of such a problem

- CH $\Rightarrow$ Yes
- MA $\Rightarrow$ Yes
- "Optimize" the above proofs to get $\mathfrak{inv} = \mathfrak{c} \Rightarrow$ YES.

Cardinal invariants of the continuum serve primarily as a scale against which we measure the complexity (or strength) of our LONG ($\mathfrak{c}$-many tasks in $\mathfrak{c}$-many steps) **recursive constructions.**

There is typically a companion SHORT ($\mathfrak{c}$-many tasks in ω_1-many steps) **recursive construction using a parametrized (weak) $\diamondsuit$-principle.**

The intention being to EITHER split the problem into manageable cases to produce a ZFC result, OR to obtain more information for the search of a suitable forcing model to prove a consistency result.

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Cardinal invariants of the continuum

"...The cardinal characteristics are simply the smallest cardinals for which various results true for $\aleph_0$ become false...."

——— A. Blass: Combinatorial Cardinal Characteristics of the Continuum

- $\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \ \forall g \in \omega^\omega \ \exists f \in \mathcal{F} \ |\{n : f(n) > g(n)\}| = \omega\}$
- $\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \ \forall A \in [\omega]^\omega \ \exists S \in \mathcal{S} \ |S \cap A| = |A \setminus S| = \omega\}$

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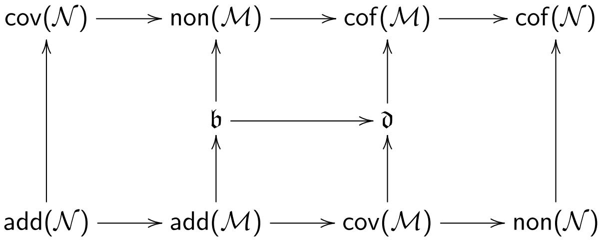
Cardinal invariants of the continuum



Cichoń's diagram

Weak diamond

Definition (Devlin-Shelah 1978)

The weak diamond principle Φ is the following assertion:

$$\forall F : 2^{<\omega_1} \to 2 \ \exists g : \omega_1 \to 2 \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary.}$$

Theorem (Devlin-Shelah 1978)

Φ is equivalent to $2^{\omega} < 2^{\omega_1}$.

Malykhin's problem

Problem (Malykhin 1978)

Is there a separable (equivalently, countable) Fréchet group which is not metrizable?

Partial positive solutions:

- $\mathfrak{p} > \omega_1$ ... Yes
- (Gerlits-Nagy 1982) There is an uncountable γ-set ... Yes
- (Nyikos 1989) $\mathfrak{p} = \mathfrak{b}$ ... Yes

Theorem (H.-Ramos García 2014)

It is consistent with **ZFC** that every separable Fréchet group is metrizable.

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Weak diamond and Fréchet groups

Theorem (H.–Ramos-García 2014)

Assuming Φ, there is a countable non-metrizable Fréchet group (of weight $\aleph_1$).

Given a filter $\mathcal{F}$ on ω let

$$\mathcal{F}^{<\omega} = \{A \subseteq [\omega]^{<\omega} : (\exists F \in \mathcal{F})[F]^{<\omega} \subseteq A\}.$$

Declaring $\mathcal{F}^{<\omega}$ the filter of neighbourhoods of the $\emptyset$ induces a group topology $\tau_{\mathcal{F}}$ on the Boolean group $[\omega]^{<\omega}$ with the symmetric difference as the group operation.

$\Phi \Rightarrow \exists$ countable non-metrizable Fréchet group

We shall use Φ to show that there is a pair of mutually orthogonal, $\subseteq^*$-increasing sequences of infinite subsets of ω (in fact, a *Hausdorff gap*) $\langle A_\alpha : \alpha < \omega_1 \rangle$, $\langle B_\alpha : \alpha < \omega_1 \rangle$ so that

for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ there exists an $\alpha < \omega_1$ such that either

1. there is an $n \in \omega$ such that $a \cap (A_\alpha \cup n) \neq \emptyset$ for every $a \in X$, or
2. for every $n \in \omega$ there is an $a \in X$ such that $\min a \geqslant n$ and $a \subset B_\alpha$.

Having done that, let $\mathcal{F}$ be the filter generated by the complements of the A_α's and the co-finite sets. Then $\tau_{\mathcal{F}}$ is Fréchet group which is not metrizable.

$\Phi \Rightarrow \exists$ countable non-metrizable Fréchet group

Recall - Φ: $\forall F : 2^{<\omega_1} \to 2$ Borel $\exists g : \omega_1 \to 2$ $\forall f \in 2^{\omega_1}$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary.}$$

Want $\langle A_\alpha : \alpha < \omega_1 \rangle$, $\langle B_\alpha : \alpha < \omega_1 \rangle$, $\subseteq^*$-increasing mutually orthogonal so that for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ there exists an $\alpha < \omega_1$ such that either

1. there is an $n \in \omega$ such that $a \cap (A_\alpha \cup n) \neq \emptyset$ for every $a \in X$, or
2. for every $n \in \omega$ there is an $a \in X$ such that $\min a \geqslant n$ and $a \subset B_\alpha$.

- The domain of F (using a suitable coding) is the set of all triples $\langle X, \langle A_\beta : \beta < \alpha\rangle, \langle B_\beta : \beta < \alpha\rangle\rangle$ such that:
  1. $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$.
  2. α is an infinite countable ordinal.
  3. $\langle A_\beta : \beta < \alpha\rangle$, $\langle B_\beta : \beta < \alpha\rangle$ is a pair of mutually orthogonal, $\subseteq^*$-increasing sequences of infinite subsets of ω.
- Given a pair $\langle A_\beta : \beta < \alpha\rangle, \langle B_\beta : \beta < \alpha\rangle$ as above, fix disjoint sets A and B such that A almost contains all A_β, $\beta < \alpha$, while B almost contains all B_β, $\beta < \alpha$, and $\omega = A \cap B$.[1]

$$F(t) = \begin{cases} 0 & \text{if } \exists n \in \omega\, \forall a \in X (a \cap (A \cup n) \neq \emptyset); \\ 1 & \text{if } \forall n \in \omega\, \exists a \in X (a \cap (A \cup n) = \emptyset). \end{cases}$$

[1] Let $\alpha = \{\alpha_n : n \in \omega\}$ be an enumeration of α. For each $n \in \omega$, let $A^{n+1} = A^n \cup \left(A_{\alpha_{n+1}} \setminus \bigcup_{k \leqslant n} B^k\right)$ and $B^{n+1} = B^n \cup \left(B_{\alpha_{n+1}} \setminus \bigcup_{k \leqslant n+1} A^k\right)$, where $A^0 = A_{\alpha_0}$ and $B^0 = B_{\alpha_0} \setminus A_{\alpha_0}$. Then, $A = \bigcup_{n\in\omega} A^n$ and $B = \omega \setminus A$ are as required.

$\Phi \;\Rightarrow\; \exists$ countable non-metrizable Fréchet group

- Now suppose that $g\colon \omega_1 \to 2$ is a $\diamondsuit$-sequence for F. Construct $\langle A_\alpha\colon \alpha < \omega_1\rangle$, $\langle B_\alpha\colon \alpha < \omega_1\rangle$ as follows:
- Let $\langle A_n\colon n < \omega\rangle$, $\langle B_n\colon n < \omega\rangle$ be any pair of mutually orthogonal, $\subseteq^*$-increasing sequences of infinite subsets of ω. If $\langle A_\beta\colon \beta < \alpha\rangle$, $\langle B_\beta\colon \beta < \alpha\rangle$ have been defined, consider the corresponding partition $\omega = A \cup B$ such that A almost contains all A_β, $\beta < \alpha$, while B almost contains all B_β, $\beta < \alpha$ constructed by the algorithm described above.
- If $g(\alpha) = 0$, then let $A_\alpha = A$, and let B_α be a co-infinite subset of B still almost containing all B_β, $\beta < \alpha$.

  If $g(\alpha) = 1$, then let $B_\alpha = B$, and let A_α be a co-infinite subset of A almost containing all A_β, $\beta < \alpha$.

Weakest weak diamond

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$$\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary (unbounded).}$$

Definition (Moore-H.-Džamonja 2004)

The weakest (or Borel) weak diamond principle $\diamondsuit(2, =)$ is the following assertion:

$$\forall F : 2^{<\omega_1} \to 2 \text{ Borel } \exists g : \omega_1 \to 2\ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary (unbounded).}^{a}$$

[a] F is Borel if $F \restriction 2^\alpha$ is Borel for every $\alpha < \omega_1$.

Borel .... $F \restriction 2^\alpha$ is Borel for every $\alpha < \omega_1$.

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Weak diamond vs. Weakest weak diamond

Theorem (Devlin-Shelah 1978)

The principle Φ holds if and only if $2^\omega < 2^{\omega_1}$. In particular, it holds assuming CH.

(Moore-H.-Džamonja 2004) $\diamondsuit(2, =)$ holds in many models of $2^\omega = 2^{\omega_1}$:

- after forcing with the Suslin tree,
- in models obtained by "definable" CS or FS iterations.

Parametrized diamonds

Definition (Moore-H.-Džamonja 2004)

The principle $\diamondsuit(\mathfrak{b})$ is the following assertion:

$$\forall \textit{Borel}\, F : 2^{<\omega_1} \to \omega^\omega \ \exists g : \omega_1 \to \omega^\omega \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \not\geq^* g(\alpha)\} \text{ is stationary.}$$

Borel .... $F \restriction 2^\alpha$ is Borel for every $\alpha < \omega_1$.

Theorem (MHD 2004)

If $\mathbb{P}_{\omega_2}$ is a CSI iteration of a sufficiently definable sufficiently homogeneous proper forcing such that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{b} = \omega_1$ then $V^{\mathbb{P}_{\omega_2}} \models \diamondsuit(\mathfrak{b})$.

Archangel'skii problem: Compact weakly first countable spaces

A topological space X is *weakly first countable* if for any point $x \in X$ there is a countable collection $\{C_n(x) : n \in \omega\}$ of subsets of X each containing x such that a set $U \subseteq X$ is open if and only if $\forall x \in U\ \exists n \in \omega\ C_n(x) \subseteq U$.

- Jakovlev 1976 (CH) There is a weakly first countable compact space which is not first countable.
- Abraham-Gorelic-Juhász 2006 ($\mathfrak{b} = \mathfrak{c}$) There is a Jakovlev space.
- Gaspar-Hernández-H. 2015 ($\diamondsuit(\mathfrak{b})$) There is a Jakovlev.

Steprāns problem: Cohen-indestructible MAD families

A *maximal almost disjoint (MAD)* family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is *Cohen-indestructible* if it remains maximal after adding a Cohen-real (equivalently, any number of Cohen-reals).

- Kunen 1980 (CH) There is a Cohen-indestructible MAD family.
- Garcia-Ferreira-H. 2001 ($\mathfrak{b} = \mathfrak{c}$) There is a Cohen-indestructible MAD family.
- Guzmán-H. 2015 ($\diamondsuit(\mathfrak{b})$) There is a Cohen-indestructible MAD family.

The Scarborough-Stone problem: Products of sequentially compact spaces

A topological space X is *sequentialy compact (resp. countably compact)* if any countable sequence in X has a convergent subsequence (resp. an accumulation point).

- Vaughan 1976 ($\diamondsuit$) There is a family of sequentially compact spaces whose product is not countably compact.
- van Douwen 1984 ($\mathfrak{b} = \mathfrak{c}$) There is a family of sequentially compact spaces whose product is not countably compact.
- Gaspar-Hernández-H. 2015 ($\diamondsuit(\mathfrak{s})$) There is a family of sequentially compact spaces whose product is not countably compact.

The principle $\diamondsuit(\mathfrak{s})$ is the following:

$$\forall \mathit{Borel} F : 2^{<\omega_1} \to [\omega]^\omega \ \exists g : \omega_1 \to [\omega]^\omega \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : g(\alpha) \text{ splits } F(f \restriction \alpha)\} \text{ is stationary.}$$

Archangel'skii-Franklin problem: Sequential order of compact spaces

Recall that a topological space X is *sequential* if any subset which is not closed contains a convergent sequence whose limit is outside of the set. In other words, closure can be obtained by iterating adding limits of convergent sequences, the *sequential order* of X being the minimal number of iterations necessary to get the closure.

- Isbell-Mrowka (implicitly) There is a compact sequential space of sequential order 2.
- Bashkirov 1974 (CH) There is a compact sequential space of sequential order ω_1.
- Dow 2005 ($\mathfrak{b} = \mathfrak{c}$) There is a compact space of sequential order 4.
- Gaspar-Henández-H. ($\diamondsuit(\mathfrak{b})$) There is a compact sequential space of sequential order ω.
- Gaspar-Henández-H. ($\diamondsuit(\mathfrak{b}\mathfrak{s})$) There is a compact sequential space of sequential order ω_1.

Parametrized weak diamonds

An invariant is a triple $(A, B, \to)$ where $\to \subseteq A \times B$ is such that
(1) $\forall a \in A\ \exists b \in B\ a \to b$, and
(2) $\forall b \in B\ \exists a \in A\ a \not\to b$.
Given an invariant $(A, B, \to)$ the evaluation of $(A, B, \to)$ is

$$||A, B, \to || = \min\{|B'| : B' \subseteq B\ \forall a \in A\ \exists b \in B'\ a \to b\}$$

We abbreviate $(A, A, \to)$ as $(A, \to)$.

Definition $\Phi(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A\ \exists g : \omega_1 \to B\ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

Disadvantage: $\Phi(A, B, \to)$ implies $2^\omega < 2^{\omega_1}$.

We restrict to Borel invariants - require A, B and $\to$ to be Borel subsets of Polish spaces.

Definition (MHD 2004) $\diamondsuit(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A \textit{ Borel } \exists g : \omega_1 \to B \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

F is Borel if $F \restriction 2^\alpha$ is Borel for every $\alpha < \omega_1$.
Easy observations:

- $\diamondsuit(A, B, \to) \ \Rightarrow \ ||A, B, \to|| \leq \omega_1$,
- $\diamondsuit \Leftrightarrow \ \diamondsuit(\mathbb{R}, =)$,
- $(A, B, \to) \leq_{GT} (A', B', \to')$ and $\diamondsuit(A', B', \to') \ \Rightarrow \ \diamondsuit(A, B, \to)$.

... and the point is ...

Theorem (MHD 2004)

If W is a canonical model and $(A, B, \to)$ is a Borel invariant then $W \models \diamondsuit(A, B, \to)$ if and only if $||A, B, \to|| \leq \omega_1$.

By a CANONICAL MODEL we mean a model which is the result of a CSI of length ω_2 of a single sufficiently definable (e.g. Suslin) and sufficiently homogeneous ($\mathbb{P} \simeq \{0,1\} \times \mathbb{P}$) proper forcing $\mathbb{P}$.

Results from (MHD)

- $\diamondsuit(\text{non}(\mathcal{M}))$ $\Rightarrow$ There is a Suslin tree.
- $\diamondsuit(\mathfrak{s}^\omega)$ $\Rightarrow$ There is an Ostaszewski space.
- $\diamondsuit(\mathfrak{b})$ $\Rightarrow$ There is a non-trivial coherent sequence on ω_1 which can not be uniformized.
- Cardinal invariants with "structure" have their Borel "shadows", e.g. $\diamondsuit(\mathfrak{b}) \Rightarrow \mathfrak{a} = \omega_1$, $\diamondsuit(\mathfrak{r}) \Rightarrow \mathfrak{u} = \omega_1, \ldots$
- CH + "Almost no diamonds hold" is consistent.

Further results

- (Yorioka, 2005) $\diamondsuit(\text{non}(\mathcal{M})) \Rightarrow$ There is a ccc destructible Hausdorff gap.
- (Minami 2005) Separated $\diamondsuit$'s for invariants in the Cichoń diagram under CH.
- (Kastermans-Zhang 2006) $\diamondsuit(\text{non}(\mathcal{M})) \Rightarrow$ There is a maximal cofinitary group of size ω_1.
- (Minami 2008) Parametrized diamonds hold in FSI iterations of Suslin ccc forcings.
- (Mildenberger, Mildenberger-Shelah 2009-2011) No other diamonds in the Cichoń diagram imply the existence of a Suslin tree (all are consistent with "all Aronszajn trees are special").

More examples

- (Cancino-H.-Meza 2014) $\diamondsuit(\mathfrak{r}) \Rightarrow$ There is a countable irresolvable space of weight ω_1.
- (H.–Ramos-García 2014) $\diamondsuit(2,=) \Rightarrow$ There is a separable Fréchet non-metrizable group.
- (Chodounský 2014) $\diamondsuit(2,=) \Rightarrow$ There is a tight Hausdorff gap of functions.
- (Fernández-H. 2015) $\diamondsuit(\mathfrak{r}_{Hindman}) \Rightarrow$ There is a union-ultrafilter of character ω_1.
- (Fernández-H. 2015) $\diamondsuit(\mathfrak{r}_{Fin\times scattered}) \Rightarrow$ There is a gruff ultrafilter of character ω_1.

Cosmetic changes

Definition $\diamondsuit(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A \text{ Borel } \exists g : \omega_1 \to B \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

It turns out that the requirement that F be Borel is unnecessarily strong – can be replaced by $F \restriction 2^\alpha$ is definable from an ω_1-sequence of reals (or even an ω_1-sequence of ordinals), i.e. $F \restriction 2^\alpha \in L(\mathbb{R})[X]$, where X is an ω_1-sequence of ordinals, which we shall call ω_1-definable.

Definition $\diamondsuit^{\omega_1}(A, B, \to)$

$$\forall F : 2^{<\omega_1} \to A \ \omega_1\text{-definable } \exists g : \omega_1 \to B \ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) \to g(\alpha)\} \text{ is stationary.}$$

The weakest weak diamond and failure of Baumgartner

$\diamondsuit^{\omega_1}(2,=)$ - the Weakest weak diamond

$$\forall F : 2^{<\omega_1} \to 2 \ \omega_1\text{-definable}\ \exists g : \omega_1 \to 2\ \forall f \in 2^{\omega_1}$$

$$\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary.}$$

Example.

$\diamondsuit^{\omega_1}(2,=) \Rightarrow$ Every $\aleph_1$-dense set of reals X contains an $\aleph_1$-dense set Y such that X and Y are not order isomorphic.

Proof.

Fix X and Z $\aleph_1$-dense subset of X such that $X \setminus Z$ is uncountable. Enumerate $X \setminus Z$ as $\{x_\alpha : \alpha < \omega_1\}$, and let $H : 2^\omega \to Aut(\mathbb{R})$ be Borel and onto. Let $F(s) = 0$ iff $|s| < \omega$ or $H(s \restriction \omega)(x_{|s|}) \in X$.

Given g, let $Y = Z \cup \{x_\alpha : g(\alpha) = 1\}$. Given an $h \in Aut(\mathbb{R})$ consider any $f \in 2^{\omega_1}$ such that $H(f \restriction \omega) = h$.

□

Sequential composition of invariants

Definition

Given $\mathfrak{i} = (A, B, \rightarrow)$ and $\mathfrak{j} = (A', B', \rightarrow')$, we define the sequential composition $\mathfrak{i};\mathfrak{j}$ of $\mathfrak{i}$ and $\mathfrak{j}$ by

$\mathfrak{i};\mathfrak{j} = (A \times A'^{B}, B \times B', \rightarrow'')$ with $(a, h) \rightarrow'' (b, b')$ iff $a \rightarrow b$ & $h(b) \rightarrow' b'$.

Remark: $||\mathfrak{i};\mathfrak{j}|| = \max\{||\mathfrak{i}||, ||\mathfrak{j}||\}$.

Maximal trees in $\mathcal{P}(\omega)/\text{fin}$.

A set $\mathcal{T} \subseteq [\omega]^\omega$ is a maximal tree if

1. $\mathcal{T}$ is a tree (ordered by reverse $\subseteq^*$), and
2. $\forall C \in [\omega]^\omega (\exists T \in \mathcal{T}$ such that $T \subseteq^* C$ or $\exists T_0, T_1 \in \mathcal{T}$ incomparable such that $C \subseteq^* T_0 \cap T_1)$.

Note that levels of the tree are incomparable families, not AD families.

(Campero-Cancino-H.-Miranda 2015)

$\diamondsuit^{\omega_1}(\mathfrak{r}_\sigma; \mathfrak{d}) \;\Rightarrow$ There is a maximal tree in $\mathcal{P}(\omega)/\text{fin}$ of size ω_1.

Question

Does every maximal tree in $\mathcal{P}(\omega)/\text{fin}$ have size at least $\mathfrak{d}$?

Further small changes - The strongest weak diamond

Definition $\diamondsuit_S^{\omega_1}(\omega_1, =)$ - the Strongest weak diamond

Let $S \subseteq \omega_1$ be stationary.

$$\forall F : 2^{<\omega_1} \to \omega_1 \ \omega_1\text{-definable } \exists g : \omega_1 \to \omega_1 \ \forall f \in 2^{\omega_1}$$

$$\{\alpha \in S : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary.}$$

Observations:

- $\diamondsuit_S^{\omega_1}(\omega_1, =) + ||A, B, \to|| \leq \omega_1 \ \Rightarrow \ \diamondsuit_S^{\omega_1}(A, B, \to)$
- $\diamondsuit_S \Leftrightarrow \mathrm{CH} + \diamondsuit_S^{\omega_1}(\omega_1, =)$.

Theorem

$\forall S \in NS(\omega_1)^+ \ \diamondsuit_S^{\omega_1}(\omega_1, =)$ holds in all canonical models.

“All” parametrized diamonds hold in the Sacks model

Theorem

$\forall S \in NS(\omega_1)^+ \; \diamondsuit_S^{\omega_1}(\omega_1, =)$ holds in any canonical model.

combined with

Theorem (Zapletal 2008)

For every Borel cardinal invariant $(A, B, \to)$ if $||A, B, \to|| < \mathfrak{c}$ can be forced then $V^{\mathbb{S}_{\omega_2}} \models ||A, B, \to|| \leq \omega_1$.

gives

Corollary

$V^{\mathbb{S}_{\omega_2}} \models \diamondsuit^{\omega_1}(A, B, \to)$ for every Borel cardinal invariant $(A, B, \to)$ such that $||A, B, \to|| \leq \omega_1$ can be forced over any model without collapsing ω_2.

Canonical models

The following hold in ALL canonical models:

- All Whitehead groups of size ω_1 are free (Shelah - $\diamondsuit_S^{\omega_1}(2,=)$)
- Baumgartner's theorem fails (Baumgartner - $\diamondsuit^{\omega_1}(2,=)$)
- $\mathfrak{p} = \mathfrak{q} = \omega_1$, $\mathfrak{a} = \mathfrak{b}$, $\mathfrak{r} = \mathfrak{u}$, $\mathfrak{s} = \mathfrak{s}_\omega$ ... (MHD)
- There is a non-metrizable separable Fréchet group.
  (H.-Ramos - $\diamondsuit(2,=)$)
- There is a Cohen indestructible MAD family.
  (H.-Guzmán - $\mathfrak{b} = \mathfrak{c} + \diamondsuit(\mathfrak{b})$)
- There is a compact sequential space of sequential order > 2.
  (Dow - $\mathfrak{b} = \mathfrak{c}$ + Gaspar-Hernández-H. - $\diamondsuit(\mathfrak{b})$)
- There is a compact weakly first countable space that is not first countable.
  (Abraham-Gorelic-Juhász - $\mathfrak{b} = \mathfrak{c}$ + Gaspar-Hernández-H. - $\diamondsuit(\mathfrak{b})$)
- There is a ccc forcing adding a real and not adding either a random or a Cohen real.
  (Brendle - $\mathrm{cof}(\mathcal{M}) = \mathfrak{c}$ + Guzmán - $\diamondsuit(\mathrm{cof}(\mathcal{M}))$).

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1. Is $\diamondsuit^{\omega_1}(\omega_1, <)$ consistent with $\neg\diamondsuit^{\omega_1}(\omega_1, =)$?
2. Does every canonical model contain a P-point?
3. Does every canonical model contain a Suslin tree?

Thank you for your attention!!!

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Thank you for your attention!!!