Interpolation sets in spaces of continuous metric-valued functions

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Joint work with Marita Ferrer and Luis Tárrega

Index

Index

Index

Index

Index

Index

Introduction: locally compact Abelian groups

Consider the group of integers $\mathbb{Z}$ and let us say that sequence (n_k) converges to 0 when the sequence (t^{n_k}) converges to 1 for all $t \in \mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$.

Question

Find a non trivial convergent sequence

Introduction: locally compact Abelian groups

Consider the group of integers $\mathbb{Z}$ and let us say that sequence (n_k) converges to 0 when the sequence (t^{n_k}) converges to 1 for all $t \in \mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$.

Question

Find a non trivial convergent sequence

Suppose that $\{n_k\}$ is a sequence which goes to 0. Then, by definition, the sequence of functions $\{t^{n_k}\}$ converges pointwise to 1 on $\mathbb{T}$. Equivalently, the sequence of functions $\{e^{i2\pi n_k x}\}$ converges pointwise to 1 in the interval $[0,1]$. Applying Lebesgue's Dominated Convergence Theorem, it follows that the sequence $\{0\} = \{\int_0^1 e^{i2\pi n_k x} dx\}$ converges to $\int_0^1 dx = 1$, which is a contradiction.

The convergence that we have considered here stems from the initial topology generated by the functions $n \longrightarrow t^n$ of $\mathbb{Z}$ into $\mathbb{T}$. This topology is called the *Bohr topology* of $\mathbb{Z}$ (denoted $\mathbb{Z}^\sharp$) and yields the largest precompact (therefore, non discrete) group topology that can be defined on the integers. Even though this topology has been widely studied recently, we are still far from understanding it well.

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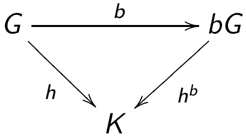
Glicksberg's Theorem

If G is a LCA group, then every Bohr compact subset of G is also compact in its original topology. This fact is due to Leptin (discrete Abelian groups) and Glicksberg (locally compact Abelian groups).

Bohr compactification

The *Bohr compactification* of an arbitrary topological group G is a pair (bG, b) where bG is a compact Hausdorff group and b is a continuous homomorphism from G onto a dense subgroup of bG with the following universal property:



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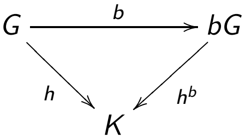

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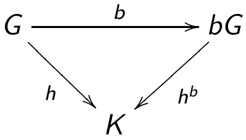

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Bohr topology

This topology is called *Bohr topology* and coincides with the weak or initial topology generated by all irreducible finite dimensional unitary representations of G. The group G equipped with the Bohr topology is denoted by $G^\sharp$.

Index

Improving the previous results considerably, van Douwen proved the following:

van Douwen's Theorem

For every infinite subset A of an Abelian group G there is $B \subset A$ with $|B| = |A|$ such that B is an interpolation set for bG. This means that B is C^*-embedded in bG, which yields $\overline{B}^{bG} \cong \beta B_d$).

These interpolation sets were called *I_0-sets* by Hartman and Ryll-Nardzewski who were the first ones to investigate them. Subsequently I_0-sets have also been called *Hartman and Ryll-Nardzewski* sets.

I_0-sets

Hadamard sets are I_0-sets

Every sequence of real numbers $\{t_n\}_{n<\omega}$ st $t_1 > 0$, $t_{n+1}/t_n > q > 1$ is an I_0-set in the additive group of the real numbers.

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Interpolation sets are well known in harmonic analysis, where they appear in connection with the Fourier transform of integrable functions and measures.

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For Abelian groups, it suffices to look at the 1-dimensional torus $\mathbb{T}$. The Bohr topology of an Abelian group G stems from the space $C_p(X, \mathbb{T})$ when X is the dual group of G.

Index

Interpolation sets in topological spaces

Definition

Let X be a topological space and let M be a metric space. A subset Y of X is called *M-interpolation* (or I_M) set when for every function $g \in M^Y$ with relatively compact range in M, there exists a $f \in C(X, M)$ such that $f_{|Y} = g$.

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Let G be a topological group and let $Hom(G, \mathbb{U}(n))$ (resp. $CHom(G, \mathbb{U}(n))$) denote the set of (resp. continuous) homomorphisms of G into $\mathbb{U}(n)$. If we equip $Hom(G, \mathbb{U}(n))$ with the pointwise convergence topology on G, it becomes a compact Hausdorff space.

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Definition

We say that a subset E of $CHom(G, \mathbb{U}(n))$ is an *n-dimensional I_0 set* when when for every bounded function $f : E \longrightarrow \mathbb{C}^{n^2}$ there exists $\widetilde{f} \in C(Hom(G, \mathbb{U}(n)), \mathbb{C}^{n^2})$ such that $\widetilde{f}|_E = f$.

Definition

Given a subset Δ of X, and $L \subseteq C(X, M)$, we say that L is *separated by Δ* if for every $F \subseteq L$, there are D_1 and D_2 subsets of M and $x \in \Delta$ such that $\mathrm{dist}(D_1, D_2) > 0$, $f(x) \in D_1$ if $f \in F$ and $f(x) \in D_2$ if $f \in L \setminus F$.

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Lemma

Let L be a countable subset of $C(X, M)$ such that $\overline{L}^{M^X}$ is a compact space. Consider the following four properties:

(a) There is a subset Δ of X such that L is separated by Δ.

(b) Every two disjoint subsets of L have disjoint closures in M^X.

(c) $\overline{L}^{M^X}$ is canonically homeomorphic to $\beta\omega$.

(d) L is a I_M set for $C_p(X, M)$.

Then $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftarrow (d)$. If M is a Banach space then (b), (c) and (d) are equivalent.

$\mathfrak{B}$-families

Definition

Let X be a topological space and let M be a metric space. We say that $G \subseteq C(X, M)$ is a *$\mathfrak{B}$-family* if the following two conditions hold:

(a) $\overline{G}^{M^X}$ is compact.

(b) There exists a nonempty open set V of X and $\epsilon > 0$ such that for every finite collection $\{U_1, \ldots, U_n\}$ of nonempty relatively open sets of V there is a $g \in G$ such that, for all $j \in \{1, \ldots, n\}$, $\mathrm{diam}(g(U_j)) \geq \epsilon$.

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Recall that a map is said *quasi open* when the closure of the image of an open subset has nonempty interior.

A technical result

Theorem

Let X be a Čech-complete space, M a metric space, Y a metrizable separable space and $\Phi : X \to Y$ a continuous and quasi-open map. If $G \subseteq C(X, M)$ is a $\mathfrak{B}$-family such that each $g \in G$ factors through Y, then there is a nonempty compact subset Δ of X and a countable infinite subset L of G such that L is separated by Δ. As a consequence, if M is a Banach space, it follows that L is an I_M set.

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The proof of this theorem proceeds by extending techniques given by Bourgain for sets of real-valued functions defined on a Polish space.

Index

Lemma

Let X be a topological group, M a metric topological group and $G \subseteq CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then G is a $\mathfrak{B}$-family if and only if it is not equicontinuous.

Topological groups

Lemma

Let X be a topological group, M a metric topological group and $G \subseteq CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then G is a $\mathfrak{B}$-family if and only if it is not equicontinuous.

Theorem

Let X be a Čech-complete group and K a compact group. If $G \subseteq CHom(X, K)$ is not equicontinuous, then G contains a countable subset L such that $\overline{L}^{K^X}$ is canonically homeomorphic to $\beta\omega$. In case $K = \mathbb{U}(n)$, it follows that L is an n-dimensional I_0 set.

Corollary

Let X be a Čech-complete Abelian group. If $G \subseteq \widehat{X}$ is not equicontinuous, then G contains an I_0 set.

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Theorem

Let X be a Čech-complete group, K be a compact group and $G \subseteq CHom(X, K)$. If for every countable subset $L \subseteq G$ and separable subset $Y \subseteq X$ it holds that either $\overline{L}^{K^Y}$ has countable tightness or $|\overline{L}^{K^Y}| \leq \mathfrak{c}$, then G is equicontinuous.

Locally k_ω abelian groups

Definition

A topological space X is a k_ω space if it is a hemicompact k-space. A topological group G is locally k_ω if G contains an open k_ω subgroup.

The class of locally k_ω abelian groups contains among others k_ω-groups, locally compact groups, and their countable direct (or inductive) limits; free abelian groups on compact spaces and, more generally, any dual group of a countable projective limit of Čech complete groups. In particular, any group that is the dual of an abelian pro-Lie group defined by a countable system is a k_ω group.

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Theorem

Every non-relatively compact subset of a locally k_ω abelian group contains an I_0-set. As a consequence every locally k_ω abelian group strongly respects compactness.

Dual space of compact non-Abelian groups

Let G be a compact group and let $\widehat{G}$ denote the set of equivalence classes of irreducible unitary representations of G.

When G is nonAbelian, $\widehat{G}$ is not longer a group and it is called *dual object*. We may view $\widehat{G}$ as a set of matrix-valued functions $\sigma : G \to \mathbb{U}(d_\sigma)$, where d_σ denotes the degree of σ.

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As in the Abelian case, one can define the notion of Fourier transform for integrable functions and measures. Given an integrable function f (resp. measure μ), the symbolism $\widehat{f}$ (resp. $\widehat{\mu}$) will denote its Fourier transform defined on $\widehat{G}$.

Interpolation sets

Let $l^\infty(\widehat{G})$, denote the Banach space of all $\{A_\sigma\}_{\sigma\in\widehat{G}}$, where A_σ is a $d_\sigma \times d_\sigma$ matrix, with norm $\|\{A_\sigma\}_{\sigma\in\widehat{G}}\|_\infty = \sup_{\sigma\in\widehat{G}} \|A_\sigma\|_{op} < \infty$ when viewed as a map on $\mathbb{C}^{d_\sigma}$. We define $l^\infty(E)$ similarly for $E \subseteq \widehat{G}$ by restricting the representations to E.

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Definition

A subset $E \subseteq \widehat{G}$ is called *Sidon set* if whenever $(A_\sigma)_\sigma \subseteq l^\infty(E)$, there is a measure μ on G satisfying $\widehat{\mu}(\sigma) = A_\sigma$ for all $\sigma \in E$. If, in addition, μ can be chosen to be discrete (or discontinuous), then E is said to be an *I_0 set*.

We finish this talk with the following result about the existence of I_0-sets, which partially extends van Douwen's theorem to nonabelian groups.

Theorem

Let G be a compact group and let L be an infinite subset of $\widehat{G}_n$. Then L contains an I_0 set.

THANK YOU FOR YOUR ATTENTION!