Compact spaces with a $\mathbb{P}$-diagonal

Tá scéilín agam

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$\mathbb{P}$-domination

Definition

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$\mathbb{P}$-domination

Definition

A space, X, is $\mathbb{P}$*-dominated* if there is a cover $\{K_f : f \in \mathbb{P}\}$ of X by compact sets such that $f \leqslant g$ (pointwise) implies $K_f \subseteq K_g$.

We call $\{K_f : f \in \mathbb{P}\}$ a $\mathbb{P}$-dominating cover.

$\mathbb{P}$-diagonal

Definition

A space, X, has a $\mathbb{P}$-*diagonal*

Definition

A space, X, has a $\mathbb{P}$-*diagonal* if the complement of the diagonal in X^2 is $\mathbb{P}$-dominated.

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Origins

Geometry of topological vector spaces (Cascales, Orihuela)

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Geometry of topological vector spaces (Cascales, Orihuela); $\mathbb{P}$-domination yields metrizability for compact subsets.

A compact space with a $\mathbb{P}$-diagonal is metrizable if it has countable tightness (no extra conditions if $\mathsf{MA}(\aleph_1)$ holds). (Cascales, Orihuela, Tkachuk).

Question

So, question: are compact spaces with $\mathbb{P}$-diagonals metrizable?

An answer

Yes if CH (Dow, Guerrero Sánchez).

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- *does not* map onto $[0,1]^{\mathfrak{c}}$

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- *does* map onto $[0,1]^{\omega_1}$

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Two important steps in that result: a compact space with a $\mathbb{P}$-diagonal

- *does not* map onto $[0,1]^{\mathfrak{c}}$, ever
- *does* map onto $[0,1]^{\omega_1}$, when it has uncountable tightness

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The answer

Theorem

Every compact space with a $\mathbb{P}$-diagonal is metrizable.

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Proof.

No compact space with a $\mathbb{P}$-diagonal maps onto $[0,1]^{\omega_1}$. □

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How does that work?

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Combinatorially

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Combinatorially: a closed set Y is BIG if there is a δ such that for every $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$ there is $y \in Y$ such that $s \subseteq y$.

A nice property of BIG sets.

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Proposition

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Proposition

A closed set is big if and only if there are a $\delta \in \omega_1$ *and* $\rho \in 2^\delta$ *such that* $\{x \in 2^{\omega_1} : \rho \subseteq x\} \subseteq Y$.

$\mathbb{P}$-domination in 2^{ω_1}

Of course 2^{ω_1} is $\mathbb{P}$-dominated: take $K_f = 2^{\omega_1}$ for all $f \in \mathbb{P}$.

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Theorem

If $\{K_f : f \in \mathbb{P}\}$ is a $\mathbb{P}$-dominating cover of 2^{ω_1} then some K_f is BIG.

The proof, case 1

$\mathfrak{d} = \aleph_1$

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$\mathfrak{d} = \aleph_1$: straightforward construction of a point not in $\bigcup_f K_f$ if we assume no K_f is BIG

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The proof, case 1

$\mathfrak{d} = \aleph_1$: straightforward construction of a point not in $\bigcup_f K_f$ if we assume no K_f is BIG, using a cofinal family of $\aleph_1$ many K_f's.

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The proof, case 3

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$\mathfrak{b} > \aleph_1$: find there are $\aleph_1$ many $s \in \mathrm{Fn}(\omega_1, 2)$ and for each there are many $h \in \mathbb{P}$ such that $s \subseteq y$ for some $y \in K_h$.

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The proof, case 3

$\mathfrak{b} > \aleph_1$: find there are $\aleph_1$ many $s \in \mathrm{Fn}(\omega_1, 2)$ and for each there are many $h \in \mathbb{P}$ such that $s \subseteq y$ for some $y \in K_h$.

We cleverly found $\aleph_1$ many h's such that each $\leqslant^*$-upper bound, f, for this family has a BIG K_f.

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The proof, case 2

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Theorem (Todorčević)

If $\mathfrak{b} = \aleph_1$ *then* 2^{ω_1} *has a subset* X *of cardinality* $\aleph_1$

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This yields another set of $\aleph_1$ many h's; the special properties of X ensure: if f is not dominated by any one of the h's then K_f is BIG.

Finishing up

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For every accumulation point, y, of $\langle y_n : n \in \omega \rangle$ we'll have

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For every accumulation point, y, of $\langle y_n : n \in \omega \rangle$ we'll have $\langle y, y \rangle \in K_f$, a contradiction.

Question

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I'd like to turn that around: does a space with a small diagonal have a $\mathbb{P}$-diagonal?
This would settle the metrizability question for spaces with a small diagonal.

Website: `fa.its.tudelft.nl/~hart`

Alan Dow and Klaas Pieter Hart,
Compact spaces with a $\mathbb{P}$-diagonal, Indagationes Mathematicae, **27** (2016), 721–726.

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