g-first countable spaces and the Axiom of Choice

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In 1966, A. Arhangel'skii introduced the notion of weak (local) base of a topological space, and consequently defined what are a g-first and g-second countable topological space.

A weak base for a topological space X is a collection $(\mathcal{W}_x)_{x\in X}$ such that $A \subseteq X$ is open if and only if for every $x \in A$, there is $W \in \mathcal{W}_x$ such that $x \in W \subseteq A$.

A topological space is *g-first countable* if it has a weak base $(\mathcal{W}_x)_{x\in X}$ such that each of the sets $\mathcal{W}_x$ is countable.

Although it looks like a first countable space is g-first countable, that is not true in the absence of some form of the Axiom of Choice.

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2. every $\mathcal{W}_x$ is a filter base;
3. $A \subseteq X$ is open if and only if
   for every $x \in A$ there is $W \in \mathcal{W}_x$ such that $x \in W \subseteq A$.

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- *g-second countable* if X has a weak base $(\mathcal{W}_x)_{x \in X}$ such that $\bigcup_{x \in X} \mathcal{W}_x$ is countable.

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Pretopological spaces can equivalently be described with neighborhoods.

$\mathcal{N}_x := \{V \mid x \notin c(X \setminus V)\}$

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$$\begin{array}{rccc} \mathcal{N}: & X & \longrightarrow & FX, \\ & x & \mapsto & \mathcal{N}_x \end{array}$$ with FX the set of filters on X.

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$$c(A) = \{x \in X \mid (\forall V \in \mathcal{N}_x)\ V \cap A \neq \emptyset\}$$

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if A is a neighborhood of all its points.

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It is clear that having a countable weak base at each point does imply being g-first countable.

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MC(α) – is **MC** for families of sets with cardinal at most α.

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C – there is $\{B(n,x) : n \in \mathbb{N}, x \in X\}$ such that for every $x \in X$, $\{B(n,x) : n \in \mathbb{N}\}$ is a local base at x.

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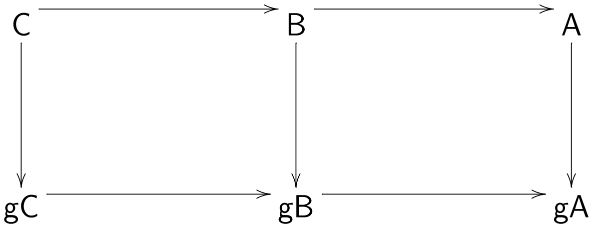
gA is never equivalent to the others.


C
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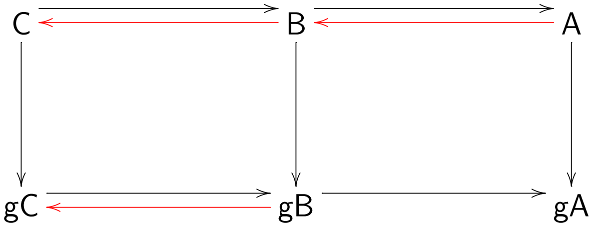
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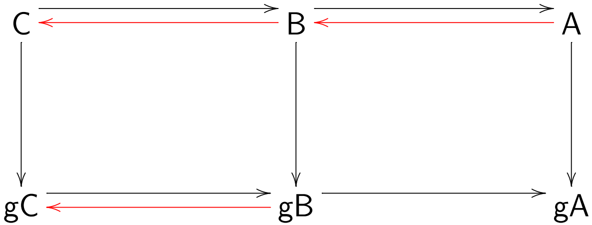

– true in ZF

– true in ZFC

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- $\mathbf{MC}(2^{\aleph_0}) \quad \Rightarrow \quad (\mathrm{gB} \Leftrightarrow \mathrm{gC}) \quad \Rightarrow \quad \mathbf{MC}(\aleph_0)$

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