Lifting homeomorphisms from separable quotients of ω^*

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Let us recall the different kinds of such automorphisms.

Clearly, every permutation of ω induces an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$.

Also, every bijection between two cofinite subsets of ω induces an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. These automorphisms are called *trivial*.

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Comfort remarked (without a proof) that it can be shown in ZFC that for any two continuous maps f and g from ω^* onto a metric space X there is a homeomorphism φ of ω^* such that $f = g \circ \varphi$.

The following theorem is an equivalent formulation of Comfort's claim:

Theorem

Let X and Y be compact metric spaces and let $f : \omega^* \to X$ and $g : \omega^* \to Y$ be continuous and onto. If $\varphi : X \to Y$ is a homeomorphism, then there is a homeomorphism $\overline{\varphi} : \omega^* \to \omega^*$ such that $\varphi \circ f = g \circ \overline{\varphi}$ and such that $\overline{\varphi}$ is induced by a permutation of ω.

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Recall that ℓ_∞ is the C^*-algebra of bounded sequences and that c_0 is the closed ideal of sequences converging to 0. The maximal ideal space of ℓ_∞/c_0 is ω^*.

Theorem (ℓ_∞/c_0-formulation)

Let $\varphi : A \to B$ be an isometric isomorphism between two separable closed subalgebras of ℓ_∞/c_0. Then φ extends to an automorphism of ℓ_∞/c_0 that is induced by a permutation of ω.

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Motivating example

Let S^1 denote the unit circle, considered as a topological group with respect to complex multiplication. Fix a countable dense subgroup $D \subseteq S^1$. Let $f : \omega \to D$ be a bijection. f has a unique continuous extension $\beta f : \beta\omega \to S^1$. Let $f^* : \omega^* \to S^1$ be the restriction of βf to ω^*. Now f^* is onto S^1.

Let $d \in D$. Multiplication by d is a homeomorphism of S^1 that is induced by a permutation of ω.

What about rotations that are not in D?

What if f is not a bijection?

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Basic observation

Lemma

The following are equivalent:

a) For every metric space X, given continuous surjections $f, g : \omega^* \to X$, there is an automorphism $\varphi : \omega^* \to \omega^*$ such that $f = g \circ \varphi$.

b) Let X be a metric space and $f : \omega^* \to X$ continuous and onto. If φ is a homeomorphism of X, then there is a homeomorphism $\overline{\varphi}$ of ω^* such that $f \circ \overline{\varphi} = \varphi \circ f$.

c) Let $f : \omega^* \to X$ and $g : \omega^* \to Y$ be continuous and onto with X and Y metric. If $\varphi : X \to Y$ is a homeomorphism, then there is a homeomorphism $\overline{\varphi}$ of ω^* such that $f \circ \overline{\varphi} = \varphi \circ g$.

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Let $f, g : \omega^* \to X$ be continuous and onto where X is metric.
We fix a countable set S_g of clopen subsets of ω^* as in the following lemma:

Lemma

Let $g : \omega^* \to X$ be onto, X a metric space.
Then there is a countable family S_g consisting of clopen subsets of ω^* such that for all continuous $f : \omega^* \to X$ and all $\varphi : \omega^* \to \omega^*$ the following holds:

If for all $A \in S_g$ we have $f[\varphi^{-1}[A]] \subseteq g[A]$, then $f = g \circ \varphi$.

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We may assume that S_g is a subalgebra of the Boolean algebra of clopen subsets of ω^*.

We construct a Boolean algebra embedding $\varphi^* : S_g \to \mathrm{clop}(\omega^*)$ such that for all clopen sets $A \in S_g$ we have $f[\varphi^*[A]] \subseteq g[A]$.

We do this inductively. Suppose we have already constructed φ^* on a finite subalgebra $\mathcal{C}$ of S_g.

We want to add a new clopen set b to $\mathcal{C}$ and extend φ^* to $\langle \mathcal{C} \cup \{e\} \rangle$, the subalgebra of S_g generated by $\mathcal{C}$ together with e.

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Theorem (Błaszczyck, Szymański)

Let X be a metric space and $f : \omega^* \to X$ onto and continuous.
Let $V \subseteq \omega^*$ be an clopen set, and suppose $C_0, C_1 \subseteq X$ are closed and such that $f[V] \subseteq C_0 \cup C_1$.
Then there is a clopen set $V_0 \subseteq V$ such that $f[V_0] = C_0 \cap f[V]$ and $f[V \setminus V_0] = C_1 \cap g[V]$.

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Now each atom of $\langle \mathcal{C} \cup \{e\} \rangle$ is of the form $a \cap e$ or $a \setminus e$ for some atom a of $\mathcal{C}$.

Using the Błaszczyk-Szymański result, we can find b such that $f[b] = g[a \cap e]$ and $f[\varphi * (a) \setminus b] = g[a \setminus e]$.

We can choose b in such a way that we can actually extend φ^* to $\langle \mathcal{C} \cup \{a \cap e\} \rangle$ by letting $\varphi^*(a \cap e) = b$.

Iterating this procedure, we can extend φ^* to all of $\langle \mathcal{C} \cup \{e\} \rangle$.

Once we have defined φ^* on $\mathcal{S}_g$, we can extend it to all of $\mathrm{clop}(\omega^*) \cong \mathcal{P}(\omega)/\mathtt{fin}$ by the result of Bella et al.

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Now each atom of $\langle \mathcal{C} \cup \{e\} \rangle$ is of the form $a \cap e$ or $a \setminus e$ for some atom a of $\mathcal{C}$.

Using the Błaszczyck-Szymański result, we can find b such that $f[b] = g[a \cap e]$ and $f[\varphi * (a) \setminus b] = g[a \setminus e]$.

We can choose b in such a way that we can actually extend φ^* to $\langle \mathcal{C} \cup \{a \cap e\} \rangle$ by letting $\varphi^*(a \cap e) = b$.

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By Stone duality there is a homeomorphism φ of ω^* such that for all clopen sets $a \subseteq \omega^*$, $\varphi^*(a) = \varphi^{-1}[a]$.

By the choice of S_g and since for all $A \in S_g$ we have $f[\varphi^{-1}[A]] = f[\varphi^*(A)] \subseteq g[A]$ it holds that $f = g \circ \varphi$.

This finishes the proof.

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Further discussion

A previous argument used the fact that for all continuous maps f from ω^* onto a metric space X there is a function $g : \omega \to X$ such that $f = g^*$.
The range of g can be any prescribed countable dense subset of X.

Based on a suggestion by K.P. Hart, using a method of Watson and Weiss we could show the following:

Theorem

There are a first countable compact space X and a continuous map f from ω^* onto X such that f is not of the form $g \restriction \omega^*$ for any $g : \beta\omega \to X$.

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Theorem (Velickovic)

$\mathrm{MA}_{\aleph_1} + \mathrm{OCA}$ *implies that all automorphisms of* $\mathcal{P}(\omega)/\mathtt{fin}$ *are trivial.*

This leads to the following observation that was pointed out by Farah.

Under $\mathrm{MA}_{\aleph_1} + \mathrm{OCA}$ there is an automorphism of a subalgebra of size $\aleph_1$ of $\mathcal{P}(\omega)/\mathtt{fin}$ that does not extend to all of $\mathcal{P}(\omega)/\mathtt{fin}$.

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The group $\mathrm{Sym}(\omega)/\mathtt{FS}$

Let $\mathrm{Sym}(\omega)$ denote the group of all permutations of ω and let $\mathtt{FS}$ denote the normal subgroup of permutations that move only finitely many elements.

Alperin, Covington, and Macpherson showed that the automorphism group of $\mathrm{Sym}(\omega)/\mathtt{FS}$ is generated by the inner automorphisms together with the automorphism induced by the shift map $n \mapsto n+1$.
It follows that the automorphism group of $\mathrm{Sym}(\omega)/\mathtt{FS}$ is isomorphic to the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$.

This leads to the following fact:

$\mathrm{Sym}(\omega)/\mathtt{FS}$ has two countable, isomorphic subgroups A and B such that no automorphism of $\mathrm{Sym}(\omega)/\mathtt{FS}$ maps A to B.

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