# $\mathcal{I}$-convergence classes

D. N. Georgiou [1], S. D. Iliadis [2], A. C. Megaritis [3] and G. A. Prinos [1]

[1] Department of Mathematics, University of Patras, Greece
[2] Department of General Topology and Geometry, Faculty of Mechanics and Mathematics, Moscow State University, Moscow 119991, Russia
[3] Technological Educational Institute of Western Greece, Greece

# Abstract

Let $X$ be a non-empty set. We consider the class $\mathcal{C}$ consisting of triads $(s, x, \mathcal{I})$, where $s = (s_d)_{d \in D}$ is a net in $X$, $x \in X$ and $\mathcal{I}$ is an ideal of $D$. We shall find several properties of $\mathcal{C}$ such that there exists a topology $\tau$ for $X$ satisfying the following equivalence: $((s_d)_{d \in D}, x, \mathcal{I}) \in \mathcal{C}$, where $\mathcal{I}$ is a proper $D$-admissible, if and only if $(s_d)_{d \in D}$ $\mathcal{I}$-converges to $x$ relative to the topology $\tau$.

# Outline

1. Preliminaries

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In this section, we recall some of the basic concepts related to the convergence of nets in topological spaces and we refer to [10] for more details.

### Ideals

Let $D$ be a non-empty set. A family $\mathcal{I}$ of subsets of $D$ is called *ideal* if $\mathcal{I}$ has the following properties:

1. $\varnothing \in \mathcal{I}$.
2. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
3. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *proper* if $D \notin \mathcal{I}$.

## Directed set

A partially ordered set $D$ is called *directed* if every two elements of $D$ have an upper bound in $D$.

If $(D, \leqslant_D)$ and $(E, \leqslant_E)$ are directed sets, then the Cartesian product $D \times E$ is directed by $\leqslant$, where $(d_1, e_1) \leqslant (d_2, e_2)$ if and only if $d_1 \leqslant_D d_2$ and $e_1 \leqslant_E e_2$. Also, if $(E_d, \leqslant_d)$ is a directed set for each $d$ in a set $D$, then the product

$$\prod_{d \in D} E_d = \{f : D \to \bigcup_{d \in D} E_d : f(d) \in E_d \text{ for all } d \in D\}$$

is directed by $\leqslant$, where $f \leqslant g$ if and only if $f(d) \leqslant_d g(d)$, for all $d \in D$.

### Net

A *net* in a set $X$ is an arbitrary function $s$ from a non-empty directed set $D$ to $X$. If $s(d) = s_d$, for all $d \in D$, then the net $s$ will be denoted by the symbol $(s_d)_{d \in D}$.

### Semisubnet

A net $(t_\lambda)_{\lambda \in \Lambda}$ in $X$ is said to be a *semisubnet* of the net $(s_d)_{d \in D}$ in $X$ if there exists a function $\varphi : \Lambda \to D$ such that $t = s \circ \varphi$. We write $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ to indicate the fact that $\varphi$ is the function mentioned above.

## Subnet

A net $(t_\lambda)_{\lambda\in\Lambda}$ in $X$ is said to be a *subnet* of the net $(s_d)_{d\in D}$ in $X$ if there exists a function $\varphi : \Lambda \to D$ with the following properties:

1. $t = s \circ \varphi$, or equivalently, $t_\lambda = s_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$.
2. For every $d \in D$ there exists $\lambda_0 \in \Lambda$ such that $\varphi(\lambda) \geqslant d$ whenever $\lambda \geqslant \lambda_0$.

## Remark

Suppose that $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ is a subnet of the net $(s_d)_{d\in D}$ in $X$. For every ideal $\mathcal{I}$ of the directed set $D$, we consider the family $\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}$. This family is an ideal of $\Lambda$ which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

### Convergence of a net

We say that a net $(s_d)_{d \in D}$ *converges* to a point $x \in X$ if for every open neighbourhood $U$ of $x$ there exists a $d_0 \in D$ such that $x \in U$ for all $d \geqslant d_0$. In this case we write $\lim_{d \in D} s_d = x$.

**$\mathcal{I}$-convergence of a net ([14])**

Let $X$ be a topological space and $\mathcal{I}$ an ideal of a directed set $D$. We say that a net $(s_d)_{d\in D}$ $\mathcal{I}$*-converges* to a point $x \in X$ if for every open neighbourhood $U$ of $x$,

$$\{d \in D : s_d \notin U\} \in \mathcal{I}.$$

In this case we write $\mathcal{I} - \lim_{d\in D} s_d = x$ and we say that $x$ is the $\mathcal{I}$*-limit* of the net $(x_d)_{d\in D}$.

If $X$ is a Hausdorff space, then a proper $\mathcal{I}$-convergent net has a unique $\mathcal{I}$-limit ([14]).

**Natural (Asymptotic) density ([8], [17])**

If $A \subseteq \mathbb{N}$, then $A(n)$ will denote the set $\{k \in A : k \leq n\}$ and $|A(n)|$ will stand for the cardinality of $A(n)$. The *natural density* of $A$ is defined by

$$\mathrm{d}(A) = \lim_{n\to\infty} \frac{|A(n)|}{n},$$

if the limit exists.

In what follows $(X, \rho)$ is a fixed metric space and $\mathcal{I}$ denotes a proper ideal of subsets of $\mathbb{N}$.

**$\mathcal{I}$-convergence of a sequence in a metric space ([12])**

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $X$ is said to be $\mathcal{I}$*-convergent* to $x \in X$ if and only if for each $\epsilon > 0$ the set $A_\epsilon = \{n \in \mathbb{N} : \rho(x_n, x) \geq \epsilon\} \in \mathcal{I}$.

**Example**

Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a proper ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence with respect to the metric $\rho$ in $X$.

**Example**

Denote by $\mathcal{I}_\mathrm{d}$ the class of all subsets $A$ of $\mathbb{N}$ with $\mathrm{d}(A) = 0$. Then $\mathcal{I}_\mathrm{d}$ is a proper ideal and $\mathcal{I}_\mathrm{d}$-convergence coincides with the statistical convergence.

Let $D$ be a directed set. For all $d \in D$ we set $M_d = \{d' \in D : d' \geq d\}$.

## $D$-admissible ideal ([14])

An ideal $\mathcal{I}$ of $D$ is called *$D$-admissible*, if $D \setminus M_d \in \mathcal{I}$, for all $d \in D$.

## Proposition ([14])

Let $X$ be a topological space, $x \in X$, and $D$ a directed set. Then,

$$\mathcal{I}_0(D) = \{A \subseteq D : A \subseteq D \setminus M_d \text{ for some } d \in D\}$$

is a proper ideal of $D$. Moreover, a net $(s_d)_{d \in D}$ converges to a point $x$ of a space $X$ if and only if $(s_d)_{d \in D}$ $\mathcal{I}_0(D)$-converges to $x$.

**Proposition** ([14, Theorem 3])

Let $X$ be a topological space and $A \subseteq X$. If the net $(s_d)_{d \in D}$ in $A$ $\mathcal{I}$-converges to the point $x \in X$, where $\mathcal{I}$ is a proper ideal of $D$, then $x \in \mathrm{Cl}_X(A)$.

In what follows $X$ is a topological space, $x \in X$, $(s_d)_{d\in D}$ is a net of $X$, and $\mathcal{I}$ is an ideal of $D$.

**Proposition**

If $(s_d)_{d\in D}$ is a net such that $s_d = x$ for every $d \in D$, then $\mathcal{I} - \lim\limits_{d\in D} s_d = x$.

**Proposition**

If $\mathcal{I}_0(D) - \lim\limits_{d\in D} s_d = x$, then for every subnet $(t_\lambda)_{\lambda\in\Lambda}$ of the net $(s_d)_{d\in D}$ we have $\mathcal{I}_0(\Lambda) - \lim\limits_{\lambda\in\Lambda} t_\lambda = x$.

**Proposition**

If $\mathcal{I} - \lim\limits_{d\in D} s_d = x$, then for every semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the net $(s_d)_{d\in D}$ we have $\mathcal{I}_\Lambda(\varphi) - \lim\limits_{\lambda\in\Lambda} t_\lambda = x$.

**Proposition**

If $\mathcal{I}-\lim_{d\in D} s_d = x$, where $\mathcal{I}$ is a proper ideal of $D$, then there exists a semisubnet $(t_\lambda)_{\lambda\in\Lambda}$ of the net $(s_d)_{d\in D}$ such that $\mathcal{I}_0(\Lambda)-\lim_{\lambda\in\Lambda} t_\lambda = x$.

**Proposition**

Let $D$ be a directed set and $\mathcal{I}$ a $D$-admissible ideal of $D$. If $(s_d)_{d\in D}$ does not $\mathcal{I}$-converge to $x$, then there exists a subnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the net $(s_d)_{d\in D}$ such that:

1. $\Lambda \subseteq D$.
2. $\varphi(\lambda) = \lambda$, for every $\lambda \in \Lambda$.
3. No semisubnet $(r_k)^{f}_{k\in K}$ of $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ $\mathcal{I}_K$-converges to $x$, for every proper ideal $\mathcal{I}_K$ of $K$.
4. $\mathcal{I}_\Lambda(\varphi)$ is a proper and $\Lambda$-admissible ideal of $\Lambda$.

## Proposition

We suppose the following:

1. $D$ is a directed set.
2. $\mathcal{I}_D$ is a proper ideal of $D$.
3. $E_d$ is a directed set for each $d \in D$.
4. $\mathcal{I}_{E_d}$ is a proper ideal of $E_d$ for each $d \in D$.
5. $\mathcal{I}_D \times \mathcal{I}_{\prod_{d \in D} E_d}$ is the family of all subsets of $D \times \prod_{d \in D} E_d$ for which: $A \in \mathcal{I}_D \times \mathcal{I}_{\prod_{d \in D} E_d}$ if and only if there exists $A_D \in \mathcal{I}_D$ such that

$$\{f(d) : (d, f) \in A\} \in \mathcal{I}_{E_d}, \text{ for each } d \in D \setminus A_D.$$

Then, the family $\mathcal{I}_D \times \mathcal{I}_{\prod_{d \in D} E_d}$ is a proper ideal of $D \times \prod_{d \in D} E_d$.

## Proposition

We suppose the following:

1. $D$ is a directed set.
2. $\mathcal{I}_D$ is a proper ideal of $D$.
3. $E_d$ is a directed set for each $d \in D$.
4. $\mathcal{I}_{E_d}$ is a proper ideal of $E_d$ for each $d \in D$.
5. $(s(d,e))_{e \in E_d}$ is a net from $E_d$ to a topological space $X$ for each $d \in D$.
6. $\mathcal{I}_D - \lim_{d \in D}(\mathcal{I}_{E_d} - \lim_{e \in E_d} s(d,e)) = x$.

Then, the net $r : D \times \prod_{d \in D} E_d \to X$, where $r(d,f) = s(d, f(d))$, for every $(d,f) \in D \times \prod_{d \in D} E_d$, $\mathcal{I}_D \times \mathcal{I}_{\prod_{d \in D} E_d}$-converges to $x$.

**$\mathcal{I}$-convergence classes**

Let $X$ be a non-empty set and let $\mathcal{C}$ be a class consisting of triads $(s, x, \mathcal{I})$, where $s = (s_d)_{d \in D}$ is a net in $X$, $x \in X$, and $\mathcal{I}$ is an ideal of $D$. We say that the net $s$ *$\mathcal{I}$-converges* ($\mathcal{C}$) to $x$ if $(s, x, \mathcal{I}) \in \mathcal{C}$. We write $\mathcal{I} - \lim_{d \in D} s_d \equiv x(\mathcal{C})$.

## $\mathcal{I}$-convergence classes

Let $X$ be a non-empty set and let $\mathcal{C}$ be a class consisting of triads $(s, x, \mathcal{I})$, where $s = (s_d)_{d \in D}$ is a net in $X$, $x \in X$ and $\mathcal{I}$ is an ideal of $D$. We say that $\mathcal{C}$ is a *$\mathcal{I}$-convergence class* for $X$ if it satisfies the following conditions:

(C1) If $(s_d)_{d \in D}$ is a net such that $s_d = x$ for every $d \in D$ and $\mathcal{I}$ is an ideal of $D$, then $\mathcal{I} - \lim\limits_{d \in D} s_d \equiv x(\mathcal{C})$.

(C2) If $\mathcal{I}_0(D) - \lim\limits_{d \in D} s_d \equiv x(\mathcal{C})$, then for every subnet $(t_\lambda)_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ we have $\mathcal{I}_0(\Lambda) - \lim\limits_{\lambda \in \Lambda} t_\lambda \equiv x(\mathcal{C})$.

(C3) If $\mathcal{I} - \lim\limits_{d \in D} s_d \equiv x(\mathcal{C})$, where $\mathcal{I}$ is an ideal of $D$, then for every semisubnet $(t_\lambda)_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ we have $\mathcal{I}_\Lambda(\varphi) - \lim\limits_{\lambda \in \Lambda} t_\lambda \equiv x(\mathcal{C})$.

## $\mathcal{I}$-convergence classes

(C4) If $\mathcal{I} - \lim\limits_{d \in D} s_d = x(\mathcal{C})$, where $\mathcal{I}$ is a proper ideal of $D$, then there exists a semisubnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ such that $\mathcal{I}_0(\Lambda) - \lim\limits_{\lambda \in \Lambda} t_\lambda = x(\mathcal{C})$.

(C5) Let $D$ be a directed set and $\mathcal{I}$ a $D$-admissible ideal of $D$. If $(s_d)_{d \in D}$ does not $\mathcal{I}$-converge $(\mathcal{C})$ to $x$, then there exists a subnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ such that:

1. $\Lambda \subseteq D$.
2. $\varphi(\lambda) = \lambda$, for every $\lambda \in \Lambda$.
3. No semisubnet $(r_k)^{f}_{k \in K}$ of $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ $\mathcal{I}_K$-converges $(\mathcal{C})$ to $x$, for every proper ideal $\mathcal{I}_K$ of $K$.
4. $\mathcal{I}_\Lambda(\varphi)$ is a proper and $\Lambda$-admissible ideal of $\Lambda$.

## $\mathcal{I}$-convergence classes

(C6) We consider the following hypotheses:

1. $D$ is a directed set.
2. $\mathcal{I}_D$ is a proper ideal of $D$.
3. $E_d$ is a directed set for each $d \in D$.
4. $\mathcal{I}_{E_d}$ is a proper ideal of $E_d$.
5. $(s(d,e))_{e \in E_d}$ is a net from $E_d$ to $X$ for each $d \in D$.
6. $\mathcal{I}_D - \lim_{d \in D} t_d \equiv x(\mathcal{C})$, where $\mathcal{I}_{E_d} - \lim_{e \in E_d} s(d,e) \equiv t_d(\mathcal{C})$, for every $d \in D$.

Then, the net $r : D \times \prod_{d \in D} E_d \to X$, where $r(d,f) = s(d, f(d))$, for every $(d,f) \in D \times \prod_{d \in D} E_d$, $\mathcal{I}_D \times \mathcal{I}_{\prod_{d \in D} E_d}$-converges $(\mathcal{C})$ to $x$.

**Theorem**

Let $\mathcal{C}$ be a $\mathcal{I}$-convergence class for a set $X$. We consider the function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathrm{cl}(A)$ is the set of all $x \in X$ such that, for some net $(s_d)_{d \in D}$ in $A$ and a proper ideal $\mathcal{I}$ of the directed set $D$, $(s_d)_{d \in D}$ $\mathcal{I}$-convergences $(\mathcal{C})$ to $x$. Then, $\mathrm{cl}$ is a closure operator on $X$ and $((s_d)_{d \in D}, x, \mathcal{I}) \in \mathcal{C}$, where $\mathcal{I}$ is a proper $D$-admissible ideal, if and only if $(s_d)_{d \in D}$ $\mathcal{I}$-converges to $x$ relative to the topology $\tau_{\mathcal{I}}$ associated with $\mathrm{cl}$.

## Convergence classes (J. Kelley)

Let $X$ be a non-empty set and let $\mathcal{C}$ be a class consisting of pairs $(s, x)$, where $s = (s_n)_{n \in D}$ is a net in $X$ and $x \in X$. We say that $\mathcal{C}$ is a *convergence class* for $X$ if it satisfies the conditions listed below. For convenience, we say that $s$ converges $(\mathcal{C})$ to $x$ or that $\lim s_n = x(\mathcal{C})$ iff $(s, x) \in \mathcal{C}$.

(C1) If $s$ is a net such that $s_n = x$ for each $n$, then $s$ converges $(\mathcal{C})$ to $x$.

(C2) If $s$ converge $(\mathcal{C})$ to $x$, then so does each subnet of $s$.

(C3) If $s$ does not converge $(\mathcal{C})$ to $x$, then there exists a subnet of $s$ no subnet of which converges $(\mathcal{C})$ to $x$.

(C4) Let $D$ be a directed set, let $E_m$ be a directed set and for each $m \in D$, let $F$ be the product $D \times \prod_{m \in D} E_m$ and for $(m, f) \in F$ let $R(m, f) = (m, f(m))$. If $\lim_m \lim_n S(m, n) = x(\mathcal{C})$, then $S \circ R$ converges $(\mathcal{C})$ to $x$.

**Theorem (J. Kelley)**

Let $(\mathcal{C})$ be a convergence class for a set $X$, and for each subset $A$ of $X$ let $\mathrm{cl}(A)$ be the set of all points $x$ such that, for some net $s$ in $A$, $s$ convergences $(\mathcal{C})$ to $x$. Then $\mathrm{cl}$ is a closure operator, and $(s, x) \in \mathcal{C}$ if and only if $s$ converges to $x$ relative to the topology $\tau$ associated with $\mathrm{cl}$.

**Problem**

Compare the above topologies $\tau_{\mathcal{I}}$ and $\tau$.

[1] G. Birkoff, *Moore-Smith convergence in general topology*, Annals of Mathematics (2) 38 (1937), no. 1, pp. 39–56.

[2] R. C. Buck, *The measure theoretic approach to density*, Amer. J. Math., 68 (1946), 560–580.

[3] R. C. Buck, *Generalized asymptotic density*, Amer. J. Math., 75 (1953), 335–346.

[4] R. Engelking, *General Topology*, Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989. viii+529 pp.

[5] H. Fast, *Sur la convergence statistique*, (French) Colloquium Math. 2 (1951), 241–244.

[6] S. P. Franklin, *Spaces in which sequences suffice*, Fundamenta Mathematicae, volume 57, (1965), pp. 107–115.

- [7] J. A. Fridy, *On statistical convergence*, Analysis 5 (1985), no. 4, 301–313.
- [8] H. Halberstam and K. F. Roth, *Sequences*. Second edition. Springer-Verlag, New York-Berlin, 1983. xviii+292 pp.
- [9] J. L. Kelley, *Convergence in topology*, Duke Math. J. 17, (1950), 277–283.
- [10] John L. Kelley, *General topology*. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.]. Graduate Texts in Mathematics, No. 27. Springer-Verlag, New York-Berlin, 1975. xiv+298 pp.
- [11] P. Kostyrko, M. Máčaj, and T. Šalát, *Statistical convergence and $\mathcal{I}$-convergence*, Real Analysis Exch., submitted.
- [12] P. Kostyrko, W. Wilczyński, and T. Šalát, *$\mathcal{I}$-convergence*, Real Anal. Exchange 26 (2000/01), no. 2, 669–685.

# Bibliography 3


[13] B. K. Lahiri and P. Das, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces*, Math. Bohemica 130 (2005), no. 2, 153–160.

[14] B. K. Lahiri and P. Das, *I and $\mathcal{I}^*$-convergence of nets*, Real Anal. Exchange 33 (2008), no. 2, 431–442.

[15] Giuseppe Di Maio and Ljubiša D. R. Kočinac, *Statistical convergence in topology*, Topology Appl. 156 (2008), no. 1, 28–45.

[16] D. S. Mitrinović, J. Sándor, and B. Crstici, *Handbook of number theory*. Mathematics and its Applications, 351. Kluwer Academic Publishers Group, Dordrecht, 1996. xxvi+622 pp.

[17] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery, *An introduction to the theory of numbers*. Fifth edition. John Wiley & Sons, Inc., New York, 1991. xiv+529 pp.

- [18] S. Pehlivan, A. Güncan, and M. A. Mamedov, *Statistical cluster points of sequences in finite dimensional spaces*, Czechoslovak Math. J. 54 (129) (2004), no. 1, 95–102.
- [19] Michael Rosen, *Number theory in function fields*. Graduate Texts in Mathematics, 210. Springer-Verlag, New York, 2002. xii+358 pp.
- [20] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca 30 (1980), no. 2, 139–150.
- [21] T. Šalát, B. C. Tripathy, and M. Ziman, *A note on $\mathcal{I}$-convergence field*, Italian J. Pure Appl. Math., 17 (2005), 45–54.
- [22] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66 (1959) 361–375.

[23] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloquium Math. 2 (1951), 73–74.

[24] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*. Third edition. Translated from the 2008 French edition by Patrick D. F. Ion. Graduate Studies in Mathematics, 163. American Mathematical Society, Providence, RI, 2015. xxiv+629 pp.

[25] J. W. Tukey, *Convergence and Uniformity in Topology*, Annals of Mathematics Studies, no. 2. Princeton University Press, Princeton, N. J., 1940. ix+90 pp.

[26] S. Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1970 xii+369 pp.

[27] A. Zygmund, *Trigonometric series*. Vol. I, II. Reprinting of the 1968 version of the second edition with Volumes I and II bound together. Cambridge University Press, Cambridge-New York-Melbourne, 1977. Vol. I: xiv+383 pp.; Vol. II: vii+364 pp.