PFA(S) implies there are many S-names

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what I mostly want to talk about today is our result

MM(S) implies that forcing with S makes all normal locally compact spaces $\aleph_1$-CWH

because the proof is so different

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from this day forward (and since a few years back)

S refers to a fixed Souslin tree in $L[A]$ that is coherent:
S is a subtree of $\omega^{<\omega_1}$ satisfying that, for each $\alpha \in \omega_1$ and $s \in S_\alpha$ and $t \in \omega^\alpha$, $t \in S$ iff $\{\beta < \alpha : t(\beta) \neq s(\beta)\}$ is finite

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Definition

A poset $\mathbb{P}$ is S-preserving means that S is still a Souslin tree in the forcing extension by $\mathbb{P}$, and, as usual

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$MA_{\mathcal{C}}(\omega_1)$ for a class $\mathcal{C}$ of posets means for $\mathbb{P} \in \mathcal{C}$: for every $\aleph_1$ many dense sets, there is a (generic) filter meeting them all.

Souslin preserving method

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where SA_{ω_1} is formulated (we could say $MA(S)$):

S is Souslin and $MA_{\mathcal{C}}(\omega_1)$ holds where $\mathcal{C}$ is all S-preserving ccc posets

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also MM(S) is investigated in Miyamoto's JSL paper

PFA(S) and PFA(S)[g] for S-generic g

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where he tells us the name of the game and a long useful list of consequences

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so Question: Does OCA imply ϕ is trivial? (is still open)

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$\exists$ LCN non $\aleph_1$-CWH   LCN implies $\aleph_1$-CWH

If $\mathbb{P}$ is proper and $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, then we want to prove that $S \Vdash p$ is $(M[g], \check{\mathbb{P}})$-generic – *same(?)* as $\mathbb{P} \times S$ is proper

because then we can apply PFA(S) to $\mathbb{P}$;

to get a useful S-name.

this is how the proof of e.g. PFA(S)$\models$ OCA works.

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If $D \in M$ is a dense subset of $\mathbb{P} \times S$, then

$$\dot{E} = \{r \in \mathbb{P} : (\exists s \in g)\ (r, s) \in D\}$$

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Let $\mathbb{P}$ be proper and $p \in \mathbb{P}$ be $(M, \mathbb{P})$-generic. Then coherence solves one problem for us, but not another.

Lemma

If $s_1, s_2 \in M \cap S$ and if s_1 forces that p is $(M[g], \check{\mathbb{P}})$-generic, then so does s_2.

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If $\dot{A} \in M$ *is any* S*-name of a subset of* $\mathbb{P}$*, then there is another* S*-name* $\dot{B} \in M$ *such that if* s_2 *is in any* S*-generic* g_2*, there is an* S*-generic* $g_1 \ni s_1$ *such that* $\dot{A}[g_2] = \dot{B}[g_1]$*. use* $\dot{A}_{s_1,s_2}$ *to denote* $\dot{B}$

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Proof.

by extending, we assume that s_1 and s_2 are in same S_β.

Simply $\dot{B} = \{(p, s_1 \oplus t) : (p, t) \in \dot{A}, s_2 \not\perp t\}$; $s_1 \oplus t = s_1 \cup (t \setminus s_2)$

and define $g_1 = \{s_1 \oplus t : s_2 \subset t \in g_2\}$. □

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$\{p_\alpha : \alpha \in \omega_1\}$ unravels as $\dot{Y} = \{(x_\delta, s_\delta) : \delta \in C\}$, a name such that

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consider e.g. $(p, s_0) \in D \in M \prec H(\theta)$ and $p = \{\ (M_1, (x_1, s_1)), (M_2, (x_2, s_2)), (M_3, (x_3, s_3))\ \}$ where $M_1 = M \cap H(\kappa)$

and as in picture for S-positions

the PFA no S-space picture

$\{x_1, x_2, x_3\}$ from a tree T

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the PFA no S-space picture

$\{x_1, x_2, x_3\}$ from a tree T

separated by models

means T is ω_1-branching

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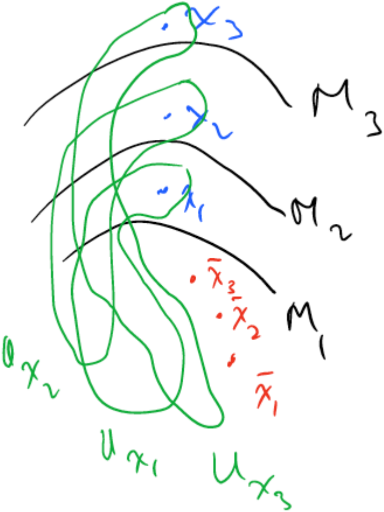
need $(M_1, \mathbb{P})$-generic

we must reflect into M_1



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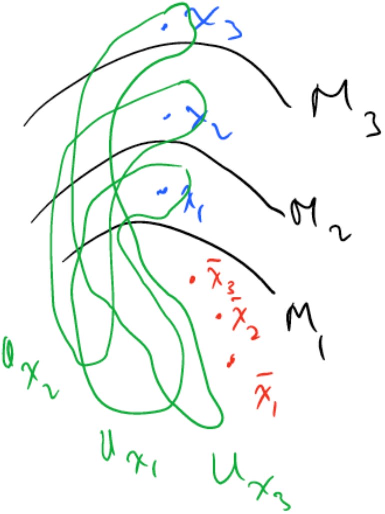

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using that $U_{x_1} \cup U_{x_2} \cup U_{x_3}$

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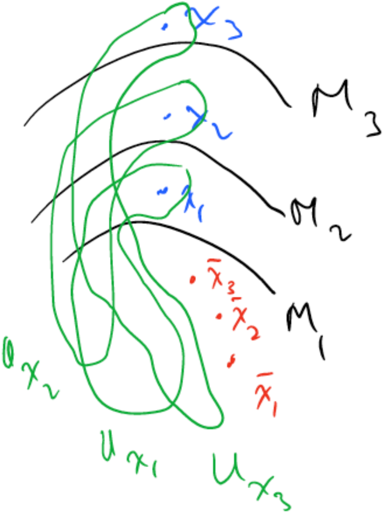
we can find $\{\bar{x}_1\} \in M_1 \cap T$



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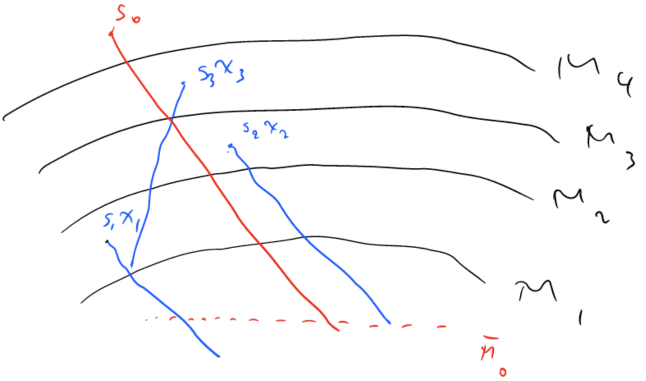
we can find $\{\bar{x}_1\} \in M_1 \cap T$

with $\bar{x}_1 \notin U_{x_1} \cup U_{x_2} \cup U_{x_3}$

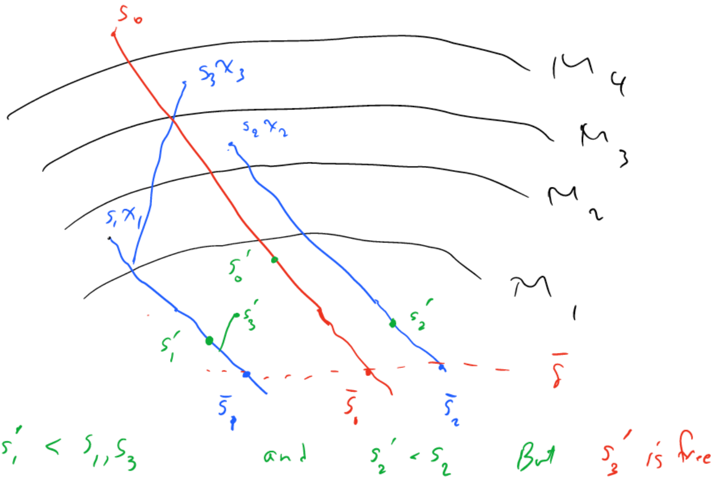
$\{x : \{\bar{x}_1, x\} \in T\} \in M_1$

repeat to find $\bar{x}_2$

now picture $s_0 \in S$ and $S \times X$ pairs from $\mathbb{P}$



s_0 is saying "you must work in my universe" but $\mathbb{P}$ has conditions as we again try to reflect into M_1



we are forced to have $s_0' < s_0$ which, by elementarity, then determines entire structure; discuss $\dot{W}_x$'s next

We set $\dot{E} = \{r : (\exists s_r)\ (r, s_r) \in D\}$ and (by $M \prec H(\theta)$) have that $s_0 \Vdash \dot{A} = \{x_1^r : r \in \dot{E}\}$ is uncountable and in M

and this is fine (enough) for finding $x_1^r \in M \setminus W_{x_1}$ such that $s_0 \Vdash x_1^r \in \dot{A}$ we can even get $x_1^r \notin W_{x_3}$

and $s_2 \Vdash (\exists r \in \dot{E} \cap M)\ \ x_2^r \notin W_{x_2}$ is (fairly) standard

but for $\dot{W}_{x_1}$, we must (and can) use that $s_1 \Vdash \dot{A}_{\bar{s}_0, \bar{s}_1}$ is uncountable

Unsolved is how to get required x_1^r not in $\dot{W}_{x_1}[s_1] \cup \dot{W}_{x_3}[s_3]$ because this may even cover $M \cap X$.

Moore-Mrowka under PFA(S)?

Remark

PFA(S) implies there is a compact sequential X that, after forcing with S, has uncountable tightness; so we can't simply apply Moore-Mrowka in PFA(S)[g] model

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thus compact countably tight X has cardinality at most $\mathfrak{c}$

MM(S)[g] $\vdash$ locally compact normal spaces are $\aleph_1$-CWH

Lemma (1. first reduction)

If all locally compact normal spaces of weight $\aleph_1$ are $\aleph_1$-CWH, then all locally compact normal spaces are $\aleph_1$-CWH.

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Forcing with S implies that the closure of a Lindelof subset of a locally compact normal space contains no uncountable closed discrete set.

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We now assume that ω_1 is a closed discrete subset of a locally compact space X of weight $\aleph_1$. We have $\{U(\alpha,\xi) : \xi \in \omega_1\}$ a neighborhood base (with compact closures) at $\alpha \in \omega_1$. For convenience $\overline{U(\alpha,\xi+1)} \subset U(\alpha,\xi) \subset U(\alpha,0)$.

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Proposition [after forcing with S]

There is a cub C_0 so that for all $\delta \in C_0$, there is a $\beta(\delta) < \delta_C^+$ such that $\omega_1 \cap \overline{\bigcup_{\alpha<\delta} U(\alpha, 0)} \subset \beta(\delta)$.

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Proof.

This is because $\overline{\bigcup_{\alpha<\delta} U(\alpha,0)}$ has a dense Lindelof subspace and so by Lemma 2, its closure has countable extent. □

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Proof.

Larson-Tall prove that in a forcing extension by S, a closed discrete set of $\aleph_1$ many points of countable character is separated if it is normalized. If A_C is not stationary, we can shrink to C_1 so that the quotient space obtained by collapsing each $Z(\alpha, C_1)$ to a point will result in the image of ω_1 being closed discrete and normalized. □

Now we use the generic g viewed as $\in \omega^{\omega_1}$

Definition

In the ground model (of MM(S)), let $\{C_\gamma : \gamma \in \omega_2\}$ be a base for the cub filter, chosen so that, for all $\zeta < \gamma$, $C_\gamma \setminus C'_\zeta$ is countable.

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Lemma

$\{E_\gamma : \gamma \in \omega_2\}$ *is a family of stationary subsets of* ω_1.

ready for the punch line

Proof.

□

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□

proof continued

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∎

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10. there is an L such that $s_\gamma \restriction \delta^+_{C_\gamma}$ forces that $\dot{U}(\bar{\beta}, 0)$ is disjoint from $\bigcup\{\dot{Z}(\alpha, C_{\zeta(\gamma)}) : \alpha < \delta \text{ and } \sigma_{C_\gamma}(\alpha) \geq L\}$ because $\alpha^+_{C_{\zeta(\gamma)}} < \delta$ for all $\alpha < \delta$ and $\dot{f}_{C_\gamma}(\dot{U}(\bar{\beta}, 0))$ is bounded.

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and, since s_{γ_L} forces that $f_{C_\gamma}(\bar{\beta}) = L$, it forces

$$\bar{\beta} \text{ is not in } \overline{\bigcup\{\dot{Z}(\alpha, C_{\zeta(\gamma_L)}) : \alpha \in \delta, \sigma_{C_\gamma}(\alpha) < L\}}.$$

□