A Generalization of the Stone Duality Theorem

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Introduction

The results which will be presented in this talk are published in ArXiv (see [DIDV]) and are submitted for a publication in another journal. They can be regarded as a natural continuation of the results from the papers [DV1], [DV3], [DV11] and, to some extent, from the papers [D-APCS09], [D2009], [D-AMH1-10], [D-AMH2-11], [D2012], [DI2016], [VDDB].

The celebrated Stone Duality Theorem ([ST], [Si]) states that the category **Bool** of all Boolean algebras and Boolean homomorphisms is dually equivalent to the category **Stone** of compact Hausdorff totally disconnected spaces and continuous maps. In this talk we will present a new duality theorem for the category of *precontact algebras* and suitable morphisms between them which implies the Stone Duality Theorem, its connected version obtained in [DV11], the recent duality theorems from [BBSV] and [GG], and some new duality theorems for the category of contact algebras and for the category of complete contact algebras.

The notion of a precontact algebra was defined independently (and in completely different forms) by S. Celani ([C]) and by I. Düntsch and D. Vakarelov ([DUV]). It arises naturally in the fields of logic, topology and theoretical computer science.

Recall that one of the central concepts in the algebraic theory of modal logic is that of modal algebra. A modal operator on a Boolean algebra B is a unary function $\Box : B \longrightarrow B$ preserving finite meets (including 1), and that a modal algebra is a pair $(B, \Box)$, where B is a Boolean algebra and $\Box$ is a modal operator on B.

S. Celani [C] generalized the concept of a modal operator to that of a *quasi-modal operator*.

The definition of quasi-modal algebras

Let $\mathcal{I}(\mathcal{B})$ be the lattice of all ideals of a Boolean algebra B. Then a *quasi-modal operator on B* is a function $\triangle : B \longrightarrow \mathcal{I}(\mathcal{B})$ preserving finite meets, and a *quasi-modal algebra* is a pair $(B, \triangle)$, where B is a Boolean algebra and $\triangle$ is a quasi-modal operator on B.

In [DUV], I. Düntsch and D. Vakarelov introduced the notion of a *proximity algebra* which is now known as a *precontact algebra* (see [DV3]). Its definition is the following:

The definition of precontact algebras

An algebraic system $\underline{B} = (B, C)$ is called a *precontact algebra* if the following holds:

- $B = (B, 0, 1, +, ., *)$ is a Boolean algebra (where the complement is denoted by "$*$");
- C is a binary relation on B (called a *precontact relation*) satisfying the following axioms:

(C0) If aCb then $a \neq 0$ and $b \neq 0$;

(C+) $aC(b+c)$ iff aCb or aCc; $(a+b)Cc$ iff aCc or bCc.

A precontact algebra (B, C) is said to be *complete* if the Boolean algebra B is complete.

The notion of precontact algebra is suitable for the purposes of theoretical computer science but can be also regarded as an algebraic generalization of the notion of proximity and thus it is interesting also for topologists.

It is easy to see that if B is a Boolean algebra then precontact relations on B are in 1-1 correspondence with quasi-modal operators on B. Indeed, for every precontact relation C on B, set $\triangle_C(a) = \{b \in B \mid b(-C)a^*\}$ for every $a \in B$. Then $\triangle_C$ is a quasi-modal operator on B. Also, for every quasi-modal operator $\triangle$ on B, set $aC_\triangle b \leftrightarrow a \notin \triangle(b^*)$ for every $a, b \in B$. Then $C_\triangle$ is a precontact relation on B. Moreover, $C_{\triangle_C} = C$ and $\triangle_{C_\triangle} = \triangle$.

In this talk, we show that there exists a duality functor G^a between the category **PCA** of all precontact algebras and suitable morphisms between them and the category **PCS** of all 2-precontact spaces and suitable morphisms between them. Then, clearly, fixing some full subcategory $\mathcal{C}$ of the category **PCA**, we obtain a duality between the categories $\mathcal{C}$ and $G^a(\mathcal{C})$. Further, taking categories which are isomorphic or equivalent to the subcategory $\mathcal{C}$ and (or) to the subcategory $G^a(\mathcal{C})$, we obtain as corollaries the Stone Duality and the other dualities mentioned above.

Preliminaries

The next definition was given in [DV1]:

Definition 1.

A precontact algebra (B, C) is called a *contact algebra* (and C is called a *contact relation*) if it satisfies the following axioms $(Cref)$ and $(Csym)$:

$(Cref)$ If $a \neq 0$ then aCa (reflexivity axiom);

$(Csym)$ If aCb then bCa (symmetry axiom).

Lemma 1.

Let (B, C) *be a precontact algebra. Define*

$$aC^{\#}b \iff ((aCb) \vee (bCa) \vee (a.b \neq 0)).$$

Then $C^{\#}$ *is a contact relation on* B *and hence* $(B, C^{\#})$ *is a contact algebra.*

Now we will give some examples of precontact and contact algebras. We will start with the *extremal contact relations*.

Example 1.

Let B be a Boolean algebra. Then there exist a largest and a smallest contact relations on B; the largest one, ρ_l (sometimes we will write ρ_l^B), is defined by

$$a\rho_l b \iff (a \neq 0 \text{ and } b \neq 0),$$

and the smallest one, ρ_s (sometimes we will write ρ_s^B), by

$$a\rho_s b \iff a.b \neq 0.$$

We are now going to show that each relational system generates canonically a precontact algebra ([DUV]).

Relational systems and precontact relations

Let (W, R) be a *relational system*, i.e. W is a non-empty set and R is a binary relation on W. Then the *precontact relation* C_R between the subsets of W is defined as follows: for every $M, N \subseteq W$,

$$MC_RN \text{ iff } (\exists x \in M)(\exists y \in N)(xRy).$$

Proposition 1. ([DUV])

Let (W, R) be a relational system and let 2^W be the Boolean algebra of all subsets of W. Then:

(a) $(2^W, C_R)$ *is a precontact algebra;*

(b) $(2^W, C_R)$ *is a contact algebra iff R is a reflexive and symmetric relation on W.*

Clearly, Proposition 1 implies that if B is a Boolean subalgebra of the Boolean algebra 2^W, then (B, C_R) is also a precontact algebra (here (and further on), for simplicity, we denote again by C_R the restriction of the relation C_R to B).

We recall as well that every topological space generates canonically a contact algebra.

The Boolean algebra of all regular closed subsets

Let X be a topological space and let $RC(X)$ be the set of all regular closed subsets of X (recall that a subset F of X is said to be *regular closed* if $F = \mathrm{cl}(\mathrm{int}(F))$). Let us equip $RC(X)$ with the following Boolean operations and *contact relation* C_X:

- $F + G = F \cup G$;
- $F^* = \mathrm{cl}(X \setminus F)$;
- $F.G = \mathrm{cl}(\mathrm{int}(F \cap G))(= (F^* \cup G^*)^*)$;
- $0 = \emptyset$, $1 = X$;
- FC_XG iff $F \cap G \neq \emptyset$.

Example 2.

Let X be a topological space. Then

$$(RC(X), C_X) = (RC(X), 0, 1, +, ., *, C_X)$$

is a complete contact algebra.

Definition 2. ([DV3,DV11])

A relational system (X, R) is called a *Stone relational space* if X is a compact Hausdorff zero-dimensional space (i.e., X is a *Stone space*) and R is a closed relation on X.

Definition 3. ([DV3,DV11])

(a) Let X be a topological space and X_0 be a dense subspace of X. Then the pair (X, X_0) is called a *topological pair*.

(b) Let (X, X_0) be a topological pair. Then we set

$$RC(X, X_0) = \{cl_X(A) \mid A \in CO(X_0)\},$$

where $CO(X)$ is the set of all clopen (=closed and open) subsets of X.

Definition 4.

Let $\underline{B} = (B, C)$ be a precontact algebra. A non-empty subset Γ of B is called a *clan* if it satisfies the following conditions:

(*Clan*1) $0 \notin \Gamma$;

(*Clan*2) If $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$;

(*Clan*3) If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$;

(*Clan*4) If $a, b \in \Gamma$ then $aC^{\#}b$.

The set of all clans of a precontact algebra $\underline{B}$ is denoted by $Clans(\underline{B})$.

Recall that a non-empty subset of a Boolean algebra B is called a *grill* if it satisfies the axioms (Clan1)-(Clan3). The set of all grills of B will be denoted by $Grills(B)$.

Notation 1.

Let $(X, \mathcal{T})$ be a topological space, X_0 be a subspace of X, $x \in X$ and B be a subalgebra of the Boolean algebra $(RC(X), +, ., *, \emptyset, X)$. We put

$$\sigma_x^B = \{F \in B \mid x \in F\};$$

and

$$\Gamma_{x,X_0} = \{F \in CO(X_0) \mid x \in cl_X(F)\}.$$

When $B = RC(X)$, we will often write simply σ_x instead of σ_x^B; in this case we will sometimes use the notation σ_x^X as well.

Definition 5. (2-PRECONTACT SPACES.)([DV3,DV11])

A triple $\underline{X} = (X, X_0, R)$ is called a *2-precontact space* if the following conditions are satisfied:

(*PCS*1) (X, X_0) is a topological pair and X is a T_0-space;

(*PCS*2) (X_0, R) is a Stone relational space;

(*PCS*3) $RC(X, X_0)$ is a closed base for X;

(*PCS*4) For every $F, G \in CO(X_0)$, $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$ implies that $F(C_R)^{\#}G$;

(*PCS*5) If $\Gamma \in Clans(CO(X_0), C_R)$ then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$.

Proposition 2. ([DV11])

If $\underline{X} = (X, X_0, R)$ is a 2-precontact space then X is a compact space.

The Main Theorem and its corollaries

Definition 6.

Let **PCA** be the category of all precontact algebras and all Boolean homomorphisms $\varphi : (B, C) \longrightarrow (B', C')$ between them such that, for all $a, b \in B$, $\varphi(a) C' \varphi(b)$ implies that aCb.
Let **PCS** be the category of all 2-precontact spaces and all continuous maps $f : (X, X_0, R) \longrightarrow (X', X'_0, R')$ between them such that $f(X_0) \subseteq X'_0$ and, for every $x, y \in X_0$, xRy implies that $f(x) R' f(y)$.

Theorem 1. (THE MAIN THEOREM: A DUALITY THEOREM FOR PRECONTACT ALGEBRAS)

The categories **PCA** *and* **PCS** *are dually equivalent.*

The duality functors from Theorem 1 will be denoted by

$$G^a : \mathbf{PCA} \longrightarrow \mathbf{PCS} \quad \text{and} \quad G^t : \mathbf{PCS} \longrightarrow \mathbf{PCA}.$$

Corollary 1. ([ST,Si])

The categories **Bool** *and* **Stone** *are dually equivalent.*

Proof. Let $\boldsymbol{B}$ be the full subcategory of the category **PCA** having as objects all (pre)contact algebras of the form (B, ρ_s^B). Then it is easy to see that the categories **Bool** and $\boldsymbol{B}$ are isomorphic. Let $\boldsymbol{S}$ be the full subcategory of the category **PCS** having as objects all 2-precontact spaces of the form (X, X, D_X), where D_X is the diagonal of X. Then it is easy to see that the categories $\boldsymbol{S}$ and **Stone** are isomorphic. It is easy to show that $G^a(\boldsymbol{B}) \subseteq \boldsymbol{S}$ and $G^t(\boldsymbol{S}) \subseteq \boldsymbol{B}$. Therefore we obtain, using Theorem 1, that the restriction of the contravariant functor G^a to the category $\boldsymbol{B}$ is a duality between the categories $\boldsymbol{B}$ and $\boldsymbol{S}$. This implies that the categories **Bool** and **Stone** are dually equivalent.

Proposition 3. ([DV3,DV11])

Let X_0 be a subspace of a topological space X. For every $F, G \in CO(X_0)$, set

$$F\delta_{(X,X_0)}G \text{ iff } cl_X(F) \cap cl_X(G) \neq \emptyset. \quad (1)$$

Then $(CO(X_0), \delta_{(X,X_0)})$ is a contact algebra.

Definition 7. (2-CONTACT SPACES.)([DV3,DV11])

A topological pair (X, X_0) is called a *2-contact space* if the following conditions are satisfied:

$(CS1)$ X is a T_0-space;

$(CS2)$ X_0 is a Stone space;

$(CS3)$ $RC(X, X_0)$ is a closed base for X;

$(CS4)$ If $\Gamma \in Clans(CO(X_0), \delta_{(X,X_0)})$ then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$.

Proposition 4. ([DV11])

If $\underline{X} = (X, X_0)$ is a 2-contact space, then X is a compact space.

Definition 8. (STONE 2-SPACES.)([DV11])

A topological pair (X, X_0) is called a *Stone 2-space* if it satisfies conditions (CS1)-(CS3) of Definition 7 and the following one:

$(S2S4)$ If $\Gamma \in Grills(CO(X_0))$ then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$.

Note that if (X, X_0) is a Stone 2-space then X is a compact connected T_0-space.

Definition 9. ([DV11])

Let (X, X_0) and (X', X'_0) be two Stone 2-spaces and $f : X \longrightarrow X'$ be a continuous map. Then f is called a *2-map* if $f(X_0) \subseteq X'_0$.
The category of all Stone 2-spaces and all 2-maps between them will be denoted by **2Stone**.

Corollary 2. ([DV11])

The categories **Bool** *and* **2Stone** *are dually equivalent.*

Proof. Let $\boldsymbol{B}'$ be the full subcategory of the category **PCA** having as objects all (pre)contact algebras of the form (B, ρ_l^B). Then it is easy to see that the categories **Bool** and $\boldsymbol{B}'$ are isomorphic. Let $\boldsymbol{S}'$ be the full subcategory of the category **PCS** having as objects all 2-precontact spaces of the form $(X, X_0, (X_0)^2)$. We show that if $(X, X_0, (X_0)^2) \in |\boldsymbol{S}'|$ then (X, X_0) is a Stone 2-space, and that the categories $\boldsymbol{S}'$ and **2Stone** are isomorphic. It is easy to see that $G^a(\boldsymbol{B}') \subseteq \boldsymbol{S}'$ and $G^t(\boldsymbol{S}') \subseteq \boldsymbol{B}'$. Therefore we obtain, using Theorem 1, that the categories **Bool** and **2Stone** are dually equivalent.

Corollary 3.

Let $\underline{X} = (X, X_0, R_1)$, $\underline{Y} = (Y, Y_0, R_2)$ *and* $\underline{X}, \underline{Y} \in |\mathbf{PCS}|$. *If* $f, g \in \mathbf{PCS}(\underline{X}, \underline{Y})$ *and* $f|_{X_0} = g|_{X_0}$ *then* $f = g$.

Corollary 4.

The category **PCS** *is equivalent to the category* **SAS** *of all Stone relational spaces and all continuous maps* $f : (X_0, R) \longrightarrow (X_0', R')$ *between them such that, for every* $x, y \in X_0$, xRy *implies* $f(x)R'f(y)$.

Proof. Let $F^t : \mathbf{PCS} \longrightarrow \mathbf{SAS}$ be the functor defined by

$$F^t(X, X_0, R) = (X_0, R)$$

on the objects of the category **PCS**, and by

$$F^t(f) = f|_{X_0}$$

for every $f \in \mathbf{PCS}((X, X_0, R), (Y, Y_0, R'))$. Then F^t is an equivalence functor.

Corollary 5. ([BBSV])

The categories **PCA** *and* **SAS** *are dually equivalent.*

Proof. Clearly, it follows from Theorem 1 and Corollary 4.

Recall that a topological space X is said to be *semiregular* if $RC(X)$ is a closed base for X.

Corollary 6.

For every Stone relational space (X_0, R) there exists a unique (up to homeomorphism) topological space X such that the triple (X, X_0, R) is a 2-precontact space (and, thus, X is a compact semiregular T_0-space).

Corollary 7. (A Duality Theorem for Contact Algebras)

The full subcategory **CA** *of the category* **PCA** *whose objects are all contact algebras is dually equivalent to the category* **CS** *of all 2-contact spaces and all continuous maps* $f:(X,X_0)\longrightarrow(X',X_0')$ *between them such that* $f(X_0)\subseteq X_0'$.

Proof. Let $\boldsymbol{S}''$ be the full subcategory of the category **PCS** whose objects are all 2-precontact spaces (X,X_0,R) for which R is a reflexive and symmetric relation. Then the categories $\boldsymbol{S}''$ and **CS** are isomorphic.
We show as well that $G^a(\mathbf{CA})\subseteq\boldsymbol{S}''$ and $G^t(\boldsymbol{S}'')\subseteq\mathbf{CA}$. Now, applying Theorem 1, we obtain that the categories **CA** and **CS** are dually equivalent.

The next corollary follows immediately from Corollary 5 and Proposition 1:

Corollary 8. ([BBSV])

The category **CA** *is dually equivalent to the full subcategory* **CSAS** *of the category* **SAS** *whose objects are all Stone relational spaces* (X, R) *such that* R *is a reflexive and symmetric relation.*

Definition 10. ([DV1])

A semiregular T_0-space X is said to be *C-semiregular* if for every clan Γ in $(RC(X), C_X)$ there exists a point $x \in X$ such that $\Gamma = \sigma_x$.

Proposition 5. ([DV1])

Every C-semiregular space X is a compact space.

Definition 11. ([DV11])

Let $(X, \mathcal{T})$ be a topological space and $x \in X$. The point x is said to be an *u-point* if for every $U, V \in \mathcal{T}$, $x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$ implies that $x \in \mathrm{cl}(U \cap V)$.

Corollary 9. (A Duality Theorem for Complete Contact Algebras)

The full subcategory **CCA** *of the category* **PCA** *whose objects are all complete contact algebras is dually equivalent to the category* **CSRS** *of all C-semiregular spaces and all continuous maps between them which preserve the u-points.*

Proof. Let $\boldsymbol{S}'''$ be the full subcategory of the category **PCS** whose objects are all 2-precontact spaces (X, X_0, R) for which R is a reflexive and symmetric relation and X_0 is extremally disconnected. Then the categories $\boldsymbol{S}'''$ and **CSRS** are isomorphic.

We show as well that

$$G^a(\mathbf{CCA}) \subseteq \boldsymbol{S}''' \text{ and } G^t(\boldsymbol{S}''') \subseteq \mathbf{CCA}.$$

Now, applying Theorem 1, we obtain that the categories **CCA** and **CSRS** are dually equivalent.

We are now going to present Corollary 7 in a form similar to that of Corollary 9. For doing this we need to recall two definitions from [GG]. We first introduce a new notion.

Definition 12.

Let X be a topological space and B be a Boolean subalgebra of the Boolean algebra $RC(X)$. Then the pair (X, B) is called a *mereotopological pair*.

Definition 13. ([GG])

(a) A mereotopological pair (X, B) is called a *mereotopological space* if B is a closed base for X. We say that (X, B) is a mereotopological T_0-space if (X, B) is a mereotopological space and X is a T_0-space.

(b) A mereotopological space (X, B) is said to be *mereocompact* if for every clan Γ of $(B, C_X \cap B^2)$ there exists a point x of X such that $\Gamma = \sigma_x^B$.

Remark 1.

(a) Obviously, if (X, B) is a mereotopological space then X is semiregular.

(b) Clearly, a space X is C-semiregular iff $(X, RC(X))$ is a mereocompact T_0-space. So that, the notion of mereocompactness is an analogue of the notion of C-semiregular space.

Having in mind the definition of an u-point of a topological space, we will now introduce the more general notion of an *u-point of a mereotopological pair*.

Definition 14.

Let (X, B) be a mereotopological pair and $x \in X$. Then the point x is said to be an *u-point of the mereotopological pair* (X, B) if, for every $F, G \in B$, $x \in F \cap G$ implies that $x \in \mathrm{cl}_X(\mathrm{int}_X(F \cap G))$.

Obviously, a point x of a topological space X is an u-point of X iff it is an u-point of the mereotopological pair $(X, RC(X))$. Also, if X is a topological space, then every point of X is an u-point of the mereotopological pair $(X, CO(X))$.

Lemma 2.

If (X, X_0) is a 2-contact space, then $(X, RC(X, X_0))$ is a mereocompact T_0-space.

Definition 15.

Let us denote by **MCS** the category whose objects are all mereocompact T_0-spaces and whose morphisms are all continuous maps between mereocompact T_0-spaces which preserve the corresponding u-points (i.e.,

$$f \in \mathbf{MCS}((X, A), (Y, B))$$

iff $f : X \longrightarrow Y$ is a continuous map and, for every u-point x of (X, A), $f(x)$ is an u-point of (Y, B)).

Theorem 2. (A DUALITY THEOREM FOR CONTACT ALGEBRAS)

The categories **CA** *and* **MCS** *are dually equivalent.*

Finally, we will show how our results imply the duality for contact algebras described in [GG].

Definition 16. ([GG])

Let **GMCS** be the category whose objects are all mereocompact T_0-spaces and whose morphisms are defined as follows:

$$f \in \mathbf{GMCS}((X, A), (Y, B))$$

iff $f : X \longrightarrow Y$ is a function such that the function

$$\psi_f : B \longrightarrow A, \quad F \mapsto f^{-1}(F),$$

is well defined and is a Boolean homomorphism.

Corollary 10. ([GG])

The categories **CA** *and* **GMCS** *are dually equivalent.*

Proof. We will derive this result from Corollary 7 showing that the categories **CS** and **GMCS** are isomorphic. Let $F^i : \mathbf{CS} \longrightarrow \mathbf{GMCS}$ be the functor defined by $F^i(X, X_0) = (X, RC(X, X_0))$ on the objects of the category **CS**, and by $F^i(f) = f$ on the morphisms of the category **CS**.

Further, let F^j : **GMCS** $\longrightarrow$ **CS** be the functor defined by $F^j(X,B) = (X, u(X,B))$, where

$$u(X,B) = \{x \in X \mid x \text{ is an u-point of } (X,B)\},$$

on the objects of the category **GMCS**, and by $F^j(f) = f$ on the morphisms of the category **GMCS**. Then $F^i \circ F^j = Id_{\mathbf{GMCS}}$ and $F^j \circ F^i = Id_{\mathbf{CS}}$. Hence, the categories **CS** and **GMCS** are isomorphic.

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Thank You!