The Zariski topology of a group

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based on joint work with D. Shakhmatov, M. Megrelishvili and D. Toller

- Markov's topologization problem and Markov's topology
- Algebraic sets and Zariski topology
- The Markov-Zariski topology of abelian groups
- The precompact Markov topology
- Markov's problem on potential density
- $\mathfrak{Z}$-Noetherian groups
- When the Zariski topology is a group topology (the Zariski topology of autohomeomorphism groups)
- Connectedness in the Zariski topology and von Neumann kernel
- On Comfort–Protasov's problem on minimally almost periodic topologizations

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Definition

A group G is topologizable if G admits a non-discrete Hausdorff group topology.

Problem 1. [Markov 1944]

Does there exist an infinite non-topologizable group ?

Markov called a subset S of G unconditionally closed, if S is closed in *every* Hausdorff group topology of G. In these terms:
G is non-topologizable iff $G \setminus \{e\}$ is unconditionally closed.

Definition (Shakhmatov-D.D. 2003)

The Markov topology $\mathfrak{M}_G$ of a group G has as closed sets precisely all unconditionally closed sets of G.

Clearly, $\mathfrak{M}_G$ is the infimum of all Hausdorff group topologies on G, so $\mathfrak{M}_G$ is T_1, all left and right shifts, as well as the inverse operation, are continuous ($\mathfrak{M}_G$ is not a group topology in general). G is non-topologizable iff $\mathfrak{M}_G$ is discrete. So to resolve Problem 1, one needs to produce an infinite group G with discrete $\mathfrak{M}_G$.

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Markov noticed that for every proper closed subgroup H of a connected Hausdorff group G has index $[G : H] = |G/H| \geq \mathfrak{c}$, as the homogeneous space G/H is Tychonov, non-trivial and connected. This is why he asked

Problem 2. [Markov 1945]

If all proper $\mathfrak{M}_G$-closed subgroups of a group G have index at least $\mathfrak{c}$, does G admit a connected Hausdorff group topology

The question was negatively answered by Pestov and by Remus for arbitrary groups. In the abelian case Markov's conditions becomes also a sufficient one:

Theorem (Shakhmatov-D.D., Adv. Math. vol. 286, 2016)

For an abelian group G the following are equivalent:
(a) G admits a connected Hausdorff group topology;
(b) all proper $\mathfrak{M}_G$-closed subgroups of a group G have index at least $\mathfrak{c}$;
(c) for every $m \in \mathbb{N}$, either $mG = \{0\}$ or $|mG| \geq \mathfrak{c}$.

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Algebraic sets

In order to approximate better the unconditionally closed sets Markov considered further properties of a subset X of a group G:

(a) *elementary algebraic* if there exist an integer $n > 0$, elements $a_1, \dots, a_n \in G$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$, such that $X = \{x \in G : x^{\varepsilon_1} a_1 x^{\varepsilon_2} a_2 \dots a_{n-1} x^{\varepsilon_n} a_n = 1\}$,

(b) *algebraic* if X is an intersection of finite unions of elementary algebraic subsets of G.

Example

Every centralizer $c_G(a) = \{x \in G : axa^{-1}x^{-1} = 1\}$ is an elementary algebraic set, so the center $Z(G)$ is an algebraic set.

Obviously, algebraic sets are unconditionally closed.
Markov proved that these two notions coincide for countable groups and asked whether this remains true in general:

Problem 3. [Markov 1944]

Are unconditionally closed sets always algebraic sets ?

A second topology may help to better handle this problem.

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Are unconditionally closed sets always algebraic sets ?

A second topology may help to better handle this problem.

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In order to approximate better the unconditionally closed sets Markov considered further properties of a subset X of a group G:

(a) *elementary algebraic* if there exist an integer $n > 0$, elements $a_1, \dots, a_n \in G$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$, such that $X = \{x \in G : x^{\varepsilon_1} a_1 x^{\varepsilon_2} a_2 \dots a_{n-1} x^{\varepsilon_n} a_n = 1\}$,

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The Zariski topology

For a group G the family $\mathfrak{A}_G$ of all algebraic sets of G contains all finite subsets of G and $\mathfrak{A}_G$ is closed under finite unions and arbitrary intersections.

Definition (Bryant 1976, Baumslag, Myasnikov and Remeslennikov)

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An uncountable group G is $\mathfrak{M}_G$-discrete whenever the following two conditions hold:

(a) there exists $m \in \mathbb{N}$ such that $A^m = G$ for every subset A of G with $|A| = |G|$;

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The Markov–Zariski topology of abelian groups

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Theorem (Shakhmatov, DD, 2006)

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Definition (Shakhmatov, DD, 2006)

For a group G define the precompact Markov topology by

$$\mathfrak{P}_G = \inf\{\text{all precompact group topologies on } G\}$$

Clearly, $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$ are T_1 topologies. Moreover, $\mathfrak{P}_G$ is discrete iff G admits no precompact group topologies.

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Example (von Neumann and Wiener: $SL_2(\mathbb{R})$ is MinAP)

Theorem (Toller-DD, Topology Appl. 2013)

Every divisible solvable non-abelian group is $\mathfrak{P}$-discrete. Hence, the Heisenberg group H_K is $\mathfrak{P}$-discrete, when K is a field with $\mathrm{char}\, K = 0$ $(\mathfrak{Z}_{H_K} < \mathfrak{P}_{H_K}$, *as* $\mathfrak{Z}_{H_K}\restriction_{Z(H_K)} = \mathfrak{Z}_{Z(H_K)}$ *is co-finite).*

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Definition (Markov, Izv. AN SSSR 1945)

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By Weyl's uniform disitribution theorem for every infinite $A = (a_n)$ in $\mathbb{Z}$ there exists $\alpha \in \mathbb{R}$ such that $(a_n\alpha)$ is uniformly distributed modulo 1, so the subset $(a_n\overline{\alpha})$ of $\mathbb{R}/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$ (so in $\langle\overline{\alpha}\rangle$ as well). Now the topology $\mathcal{T}$ on $\mathbb{Z}$ induced by $\mathbb{Z} \cong \overline{\alpha} \hookrightarrow \mathbb{R}/\mathbb{Z}$ works.

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Characterize the potentially dense subsets of an abelian group.

A hint. [two necessary conditions]

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If an Abelian group with $|G| \leq \mathfrak{c}$ is either torsion-free or has exponent p, then every infinite set of G is potentially dense.

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Can this be extended to groups with $|G| \leq 2^{\mathfrak{c}}$?

The answer is (more than) positive:

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For a countably infinite subset A of an Abelian group G TFAE:
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A topological space X is Noetherian, if X satisfies the ascending chain condition on open sets (or, equivalently, the minimal condition on closed sets). Actually, a space is Noetherian iff all its subspaces are compact (so an infinite Noetherian spaces are never Hausdorff).

Linear groups are $\mathfrak{Z}$-Noetherian, as their topology is coarser than the affine Zariski topology.

Theorem (Toller - DD, 2012)

A group G is $\mathfrak{Z}$-Noetherian iff every countable subgroup of G is $\mathfrak{Z}$-Noetherian.

Since countable free groups are linear, one obtain to following known result:

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Every free group is $\mathfrak{Z}$-Noetherian.

Since every group is a quotient of a free group, this shows that the quotient of a $\mathfrak{Z}$-Noetherian group need not be $\mathfrak{Z}$-Noetherian.

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Every free group is $\mathfrak{Z}$-Noetherian.

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Theorem (Toller - DD, 2012)

If N is a $\mathfrak{Z}_G$-closed normal subgroup of a $\mathfrak{Z}$-Noetherian (resp. $\mathfrak{Z}$-compact) group G, then also the quotient group G/N is $\mathfrak{Z}$-Noetherian (resp. $\mathfrak{Z}$-compact).

For a direct product $G = \prod_{i \in I} G_i$, one has $\mathfrak{Z}_G \leq \prod_{i \in I} \mathfrak{Z}_{G_i}$. These two topologies need not coincide. This inclusion implies that direct products of $\mathfrak{Z}$-compact groups are $\mathfrak{Z}$-compact.
Bryant proved that a finite product $G = G_1 \times \ldots \times G_n$ is $\mathfrak{Z}$-Noetherian if and only if every G_i is $\mathfrak{Z}$-Noetherian. This can be extended to infinite products and direct sums as follows:

Theorem (Toller - DD, 2012)

Let $\{G_i \mid i \in I\}$ be a non-empty family of groups, $G = \prod_{i \in I} G_i$ and $S = \bigoplus_{i \in I} G_i$. Then the following conditions are equivalent.

(i) *every G_i is $\mathfrak{Z}$-Noetherian and all but finitely many of the groups G_i are abelian.*

(ii) *G is $\mathfrak{Z}$-Noetherian.*

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$\mathfrak{Z}$-Hausdorff and $\mathfrak{M}$-Hausdroff groups

If $\{F_i \mid i \in I\}$ is a family of finite groups, and $G = \prod_{i \in I} F_i$, then the product topology $\prod_{i \in I} \mathfrak{Z}_{F_i}$ is a compact Hausdorff group topology, so

$$\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G \subseteq \prod_{i \in I} \mathfrak{Z}_{F_i}.$$

(a) G is $\mathfrak{Z}$-Hausdorff if and only if $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \prod_{i \in I} \mathfrak{Z}_{F_i}$.
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Example (Toller, DD 2012)

If $\{F_i \mid i \in I\}$ is a non-empty family of finite center-free groups, and $G = \prod_{i \in I} F_i$, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \prod_{i \in I} \mathfrak{Z}_{F_i}$ is a Hausdorff group topology on G, so G is $\mathfrak{Z}$-Hausdorff and $\mathfrak{Z}$-compact.

Theorem (Gaughan, Proc. Nat. Acad. Sci. USA 1967)

For the permutation group $S(X)$ of an infinite set X, the point-wise convergence topology $\mathcal{T}_p$ of $S(X)$ coincides with $\mathfrak{M}_{S(X)}$. So, $S(X)$ is $\mathfrak{M}$-Hausdorff.

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When $\mathfrak{Z}_G$ is a group topology

Here we consider a stronger condition on $\mathfrak{Z}_G$ and $\mathfrak{M}_G$, that ensures $\mathfrak{Z}$-Hausdorffness and $\mathfrak{M}$-Hausdorffness, resp.

Call a group G a $\mathfrak{Z}$-group ($\mathfrak{M}$-group), if $\mathfrak{Z}_G$ ($\mathfrak{M}_G$, resp.) is a group topology. Clearly, $\mathfrak{Z}$-groups are also $\mathfrak{M}$-groups.

The existence of an $\mathfrak{M}$-group G that it is not a $\mathfrak{Z}$-group (so $\mathfrak{Z}_G \neq \mathfrak{M}_G$) will provide a counterexample to Markov's problem 3.

Definition (Doïtchinov/Choquet, Stephenson, Jr.; Gartside-Glyn)

A Hausdorff topological group (G, τ) is called

- minimal if its topology cannot be properly weakened to another Hausdorff group topology.
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(Minimum topologies are called also a-minimal by Megrelishvli-DD [RPGT3] or Toller-DD [2012].)

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According to Gaughan's Theorem, for an infinite set X, $\mathcal{T}_p$ is a minimum topology on $S(X)$ (so, $S(X)$ is an $\mathfrak{M}$-group).

For a set X, let $S_\omega(X)$ be the subgroup of all $f \in S(X)$ with finite support $supp(f) = \{x \in X : f(x) \neq x\}$.

Confirming simultaneously a conjecture from Prodanov, Stoyanov, DD [Topological groups, 1989] as well as a conjecture of Shakhmatov, DD [OPiT2, 2006] Banakh, Guran and Protasov, improved significantly Gaughan's theorem:

Theorem (Banakh, Guran and Protasov, Topology Appl. 2012)

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In the sequel, $\mathcal{H}(X)$ denotes the autohomeomorphism group of a (compact) space X and τ_k its compact-open topology. For a (discrete) set X and its one-point compactification αX the groups $(\mathcal{H}(\alpha X), \tau_k)$ and $(S(X), \mathcal{T}_p)$ are topologically isomorphic. This fact and the above theorem suggest to investigate when the groups $\mathcal{H}(X)$ are $\mathfrak{Z}$-groups.

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When the groups $\mathcal{H}(X)$ are $\mathfrak{Z}$-groups

Theorem (Gartside-Glyn Topology Appl., 2003)

$\mathfrak{Z}_{\mathcal{H}(M)} = \tau_k$ for any metric one-dimensional manifold (with or without non-trivial boundary) M (so $\mathcal{H}(M)$ is a $\mathfrak{Z}$-group).

Motivated by this theorem and a recent theorem of van Mill (*for $n \geq 1$, $\mathcal{H}(M_n)$ is not minimal, for the n-dimensional universal Menger continuum M_n*), Megrelishvili and Polev proved:

Theorem (Megrelishvili-Polev, Topology Appl., 2016)

If X is a compact connected ordered space such that for every pair $a < b$ in X there exist $c, d \in X$ with $a \leq c < d \leq b$ and $[c, d]$ separable, then $\mathfrak{Z}_{\mathcal{H}(X)} = \tau_k$ (so, $\mathcal{H}(X)$ is a $\mathfrak{Z}$-group).

As X one can take for example $[0, 1]^2$ with the lexicographic order, or the extended long line.
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For a space X let

$$O_X = \{x \in X : x \text{ has a neighborhood homeomorphic to } (0,1)\},$$

("the one-manifold part of X"),

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More precisely, one can prove that

(a) $\mathfrak{Z}_G \neq \tau_k$, if either $|\overline{S}_X \cap C_X| \geq 2$ or there is a component I of I_X with $|\overline{I} \cap C_X| \geq 3$;

(b) $\mathfrak{Z}_G = \tau_k$, if one of the following holds:

- (b$_1$) $\dim C_X = 0$, $X \setminus S_X$ has only finitely many components, and S_X has at most one limit point in C, or
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Example (Chang and Gartside 2016)

For the autohomeomorphism group G of the topologist's sine curve $\mathfrak{Z}_G = \tau_k^* < \tau_k$ is Hausdorff (so G is an $\mathfrak{Z}$-group).

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Gabriyelyan proved that all countable unbounded groups admit a MinAP topology, resolving the second part of Problem $\mathbb{C}$. He obtained these results as particular cases when trying to resolve the more general question of describing all subgroups H of a given abelian group G such that there exists a Hausdorff group topology τ on G with $n(G,\tau) = H$. This justifies the following definition:

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Let H be a subgroup of an abelian group G. We say that H is a *potential von Neumann kernel* of G, if there exists a Hausdorff group topology τ on G such that $n(G,\tau) = H$.

In these terms the above "realization problem" for the von Neumann kernel $n(G)$ can be formulated as follows:

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Gabriyelyan resolved Problem $\mathbb{G}$ for "small" subgroups H (i.e., either countable or bounded):

Theorem (Gabriyelyan, Topology Appl. 2014)

A subgroup H of an abelian group G is a potential von Neumann kernel of G if one of the following conditions holds:

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This gives a solution to Problem $\mathbb{C}$ when G itself is "small" in the above sense:

Corollary (Gabriyelyan, Proc. Amer. Math. Soc. 2015)

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Zariski topology detects potential von Neumann kernels

The following easy lemma is helpful for finding a necessary condition that all potential von Neumann kernels must satisfy.

Lemma (Shakhmatov-DD. 2014)

The von Neumann kernel of a topological group G is contained in every open subgroup of G and contains every minimally almost periodic subgroup of G.

Proof.

If H is an open subgroup of G, then G/H is discrete, so it is maximally almost periodic. Since the characters of G/H separate points of G/H, we get $n(G) \subseteq H$. The last assertion is clear. □

Corollary

If H is an open MinAP subgroup of a topological abelian group G, then $H = n(G)$.

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Necessary conditions for the existence of a MinAP topology

Lemma (Necessary condition for potential von Neumann kernels)

All potential von Neumann kernels of an abelian group G are contained in $c_3(G)$.

Proof. Indeed, if H is a potential von Neumann kernel witnessed by some Hausdorff group topology τ with $H = n(G, \tau)$, then $c_3(G)$ being an unconditionally closed subgroup of G of finite index is τ-open, so $H \leq c_3(G)$ by the above lemma. □

By taking $H = G$ in this lemma, one obtains the following necessary condition for the existence of a MinAP topology on arbitrary abelian groups (to be compared with Problem $\mathbb{C}$).

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If an abelian group G admits a MinAP topology, then G is $\mathfrak{Z}_G$-connected.

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All potential von Neumann kernels of an abelian group G are contained in $\mathfrak{c}_3(G)$.

Proof. Indeed, if H is a potential von Neumann kernel witnessed by some Hausdorff group topology τ with $H = n(G, \tau)$, then $\mathfrak{c}_3(G)$ being an unconditionally closed subgroup of G of finite index is τ-open, so $H \leq \mathfrak{c}_3(G)$ by the above lemma. $\square$

By taking $H = G$ in this lemma, one obtains the following necessary condition for the existence of a MinAP topology on arbitrary abelian groups (to be compared with Problem $\mathbb{C}$).

Corollary

If an abelian group G admits a MinAP topology, then G is $\mathfrak{Z}_G$-connected.

We show that surprisingly, this quite simple necessary conditions is also sufficient for the existence of a MinAP topology.

Theorem (Main Theorem, Shakhmatov-DD. 2014)

For an abelian group an abelian group G, the following are equivalent:

(a) *G admits a MinAP group topology;*

(b) *G is $\mathfrak{Z}$-connected;*

(c) *all proper unconditionally closed subgroups of G have infinite index;*

(d) *for every $m \in \mathbb{N}$, either $mG = \{0\}$ or mG is infinite.*

Since unbounded groups are $\mathfrak{Z}$-connected, we obtain as immediate corollary a complete solution of Problem $\mathbb{C}$:

Corollary

Every unbounded abelian group admits a MinAP topology.

As another corollary we obtain also complete solution of Problem $\mathbb{G}$:

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A subgroup H of an abelian group G is a potential von Neumann kernel iff $H \leq c_3(G)$.

Proof. The necessity was proved above. To prove the sufficiency, assume that $H \subseteq c_3(G)$ and consider two cases.

Case 1. H is bounded. If G is unbounded, then H is a potential von Neumann kernel by Gabiyelyan's theorem.

Suppose now that G itself is bounded. Since $H \subseteq c_3(G)$ by our assumption, and $c_3(G) = G[m]$ (with $m = eo(G)$), so G contains $\bigoplus_\omega \mathbb{Z}(m)$ (Shakh.DD [2010]). As $mH = 0$, $k = o(H)$ divides m, so G contains $\bigoplus_\omega \mathbb{Z}(k)$. Now H is a potential von Neumann kernel of G again by Gabiyelyan's theorem.

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Corollary

If an abelian group admits a connected group topology, then it admits also a MinAP topology.

Indeed, if τ is a connected Hausdorff group topology on G, then $\tau \geq \mathfrak{M}_G = \mathfrak{Z}_G$, so G is also $\mathfrak{Z}$-connected, so admits a MinAP topology.

Finally, a few words about the origin of Problem $\mathbb{G}$. In analogy to the obvious fact that $G/n(G)$ is MAP, one may expect that the von Neumann kernel $n(G)$ is necessarily MinAP (i.e., $n(n(G)) = n(G)$).

Problem $\mathbb{L}$ [Gábor Lukásc 2004, unpublished]

Is the subgroup $n(G)$ of a topological abelian group always MinAP?

Lukásc [2006] built examples of group topologies on $G = \mathbb{Z}(p^\infty)$ having finite but non-trivial $n(G)$, so clearly $n(n(G)) = 0 \neq n(G)$. He asked for a description of the abelian groups that admit a group topology τ such that $n(G,\tau) \neq 0$ is finite. Partial results were obtained by him and by Nguyen [2009]. The final solution was given by Gabriyelyan [2009]. This triggered Problem $\mathbb{G}$.

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We are left with the proof of the Main Theorem.
To formulate the idea of the proof we need to recall some properties of the class MinAP.

(i) If H is a dense subgroup of an abelian topological group G, then H is MinAP iff G is MinAP.

(ii) If $\{G_i : i \in I\}$ is a family of MinAP groups, then also $\prod_i G_i$ is MinAP.

(iii) (3-space property) If G has a closed MinAP subgroup with MinAP quotient G/N, then G is MinAP.

From (i) and (ii) we deduce that if K is a MinAP group, such that an abelian group G densely embeds into some power of K, then this embedding will induce on G a MinAP topology.
The problem is to find a MinAP group K such that every abelian group G satisfying the necessary conditions (b)-(d) from the Main Theorem densely embeds into some power of K. Such a group K can be obtained as a special case of a general construction proposed by S. Hartman and J. Mycielski in 1958.

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(ii) If $\{G_i : i \in I\}$ is a family of MinAP groups, then also $\prod_i G_i$ is MinAP.

(iii) (3-space property) If G has a closed MinAP subgroup with MinAP quotient G/N, then G is MinAP.

From (i) and (ii) we deduce that if K is a MinAP group, such that an abelian group G densely embeds into some power of K, then this embedding will induce on G a MinAP topology.
The problem is to find a MinAP group K such that every abelian group G satisfying the necessary conditions (b)-(d) from the Main Theorem densely embeds into some power of K. Such a group K can be obtained as a special case of a general construction proposed by S. Hartman and J. Mycielski in 1958.

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