Ideal quasi-normal convergence and related notions

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Basic notions

- ***Ideal***: A hereditary family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ ($B \in \mathcal{I}$ for any $B \subseteq A \in \mathcal{I}$) that is closed under unions ($A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$), contains all finite subsets of ω and $\omega \notin \mathcal{I}$.

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- ***Filter:*** For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote $\mathcal{A}^d = \{\omega \setminus A : A \in \mathcal{A}\}$. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a **filter** if $\mathcal{F}^d$ is an ideal.

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- ***Associated Filter:*** If $\mathcal{I}$ is a proper ideal in Y (i.e. $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a *filter* in Y.

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  It is called the filter associated with the ideal $\mathcal{I}$.
  • If $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal then $\mathcal{B} \subseteq \mathcal{I}$ is a **base** of $\mathcal{I}$ if for any $A \in \mathcal{I}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$. We recall a folklore fact: the family of all finite intersections of elements of a family $\mathcal{A} \subseteq [\omega]^\omega$ is a base of some filter if and only if $\mathcal{A}$ has the **finite intersection property**, shortly **f.i.p.**.

- ***cof($\mathcal{I}$):*** For an ideal $\mathcal{I}$ we denote

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- ***almost contained:*** A set A is **almost contained** in a set B, written $A \subseteq^* B$, if $A \setminus B$ is finite. Assume that $\mathcal{A} \subseteq \mathcal{I}$ is such that every $B \in \mathcal{I}$ is almost contained in some $A \in \mathcal{A}$. Then $\mathcal{B} = \{A \cup F : A \in \mathcal{A} \wedge F \in [\omega]^{<\omega}\}$ is a base of $\mathcal{I}$. Moreover, if $\mathcal{A}$ is infinite, then $|\mathcal{B}| = |\mathcal{A}|$.

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- ***P-ideal:*** An ideal $\mathcal{I}$ is said to be a P-**ideal**, if for any countable $\mathcal{A} \subseteq \mathcal{I}$ there exists a set $B \in \mathcal{I}$ such that $A \subseteq^* B$ for each $A \in \mathcal{A}$. Some authors say that $\mathcal{I}$ satisfies the property (AP). If $A \subseteq \omega$ is such that $\omega \setminus A$ is infinite, then

$$\langle A \rangle^* = \{B \subseteq \omega : B \subseteq^* A\}$$

is a P-ideal with a countable base.

- ***pseudointersection:*** An infinite set $B \subseteq \omega$ is said to be a **pseudointersection** of a family $\mathcal{A} \subseteq [\omega]^\omega$ if $B \subseteq^* A$ for any $A \in \mathcal{A}$. We can introduce the dual notion: a set B is a **pseudounion** of the family $\mathcal{A}$ if $\omega \setminus B$ is infinite and if $A \subseteq^* B$ for any $A \in \mathcal{A}$. Thus an ideal $\mathcal{I}$ is P-ideal if and only if every countable subfamily of $\mathcal{I}$ has a pseudounion belonging to $\mathcal{I}$. If a pseudounion A of $\mathcal{I}$ belongs to $\mathcal{I}$, then $\mathcal{I} = \langle A \rangle^*$.

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- ***Tall ideal:*** An ideal $\mathcal{I}$ is **tall**, if for any $B \in [\omega]^\omega$, there exists an $A \in \mathcal{I}$ such that $A \cap B$ is infinite. Thus, an ideal $\mathcal{I}$ has a pseudounion if and only if $\mathcal{I}$ is not tall.

- ***pseudointersection number:*** The **pseudointersection number** is the cardinal

  $$\mathfrak{p} = \min\{|\mathcal{A}| : (\mathcal{A} \subseteq [\omega]^\omega \text{ has f.i.p. and has no pseudointersection})$$

  Thus, if $\mathcal{I}$ is an ideal with $\mathrm{cof}(\mathcal{I}) < \mathfrak{p}$, then $\mathcal{I}$ has a pseudounion. Since $\mathfrak{p} > \aleph_0$, any ideal with a countable base has a pseudounion.

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  Thus, if $\mathcal{I}$ is an ideal with $\mathrm{cof}(\mathcal{I}) < \mathfrak{p}$, then $\mathcal{I}$ has a pseudounion. Since $\mathfrak{p} > \aleph_0$, any ideal with a countable base has a pseudounion.
  • An ideal $\mathcal{I}$ with a countable base can be constructed with a pseudounion such that no pseudounion of $\mathcal{I}$ belongs to $\mathcal{I}$ and such that $\mathcal{I}$ is not a P-ideal. Assuming $\mathfrak{p} > \aleph_1$, one can construct a P-ideal $\mathcal{I}$ with an uncountable base of cardinality $< \mathfrak{p}$ such that no pseudounion of $\mathcal{I}$ belongs to $\mathcal{I}$.

- ***Ideal convergence:*** A sequence $\langle x_n : n \in \omega \rangle$ of elements of a topological space X $\mathcal{I}$-**converges** to $x \in X$, written $x_n \xrightarrow{\mathcal{I}} x$, if for each neighborhood U of x, the set $\{n \in \omega : x_n \notin U\} \in \mathcal{I}$, i.e., if the function $\langle x_n : n \in \omega \rangle$ from ω into X converges modulo filter $\mathcal{I}^d$ to x in the sense of H. Cartan.

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- ***Ideal divergence:*** A sequence $\langle x_n : n \in \omega \rangle$ is $\mathcal{I}$**-divergent to** ∞, written $x_n \xrightarrow{\mathcal{I}} \infty$, if $\{n : x_n < a\} \in \mathcal{I}$ for any positive real $a > 0$.

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- ***Ideal divergence:*** A sequence $\langle x_n : n \in \omega \rangle$ is $\mathcal{I}$**-divergent to** ∞, written $x_n \xrightarrow{\mathcal{I}} \infty$, if $\{n : x_n < a\} \in \mathcal{I}$ for any positive real $a > 0$.

  • By function f, we always mean a **real function** defined on X. A sequence of real functions $\langle f_n : n \in \omega \rangle$ $\mathcal{I}$-**converges to a real function** f on X, written $f_n \xrightarrow{\mathcal{I}} f$, if $f_n(x) \xrightarrow{\mathcal{I}} f(x)$ for each $x \in X$.

- ***$\mathcal{I}$- Quasinormal convergence:*** A sequence of real functions $\langle f_n : n \in \omega \rangle$ on X $\mathcal{I}$-**quasi-normally converges** to a real function f on X, shortly $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ on X, if there exists a sequence of reals $\langle \varepsilon_n : n \in \omega \rangle$ that $\mathcal{I}$-converges to 0 (the **control sequence**) and such that $\{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for any $x \in X$.

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- ***strongly $\mathcal{I}$-quasi-normal convergence:*** We say that a sequence **strongly $\mathcal{I}$-quasi-normally converges** to f if the control sequence is $\langle 2^{-n} : n \in \omega \rangle$. We write $f_n \stackrel{\mathrm{s}\mathcal{I}\mathrm{QN}}{\longrightarrow} f$. In fact one can replace the sequence $\langle 2^{-n} : n \in \omega \rangle$ by any sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$.

- $\mathcal{I}$ ***- Uniform Convergence*** A sequence of real functions $\langle f_n : n \in \omega \rangle$ on X $\mathcal{I}$-**uniformly converges** to a real function f, shortly $f_n \xrightarrow{\mathcal{I}\text{-u}} f$, if there exists a set $A \in \mathcal{I}$ such that $\{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon\} \subseteq A$ for any $\varepsilon > 0$ and any $x \in X$.

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  • Evidently the notion of $\mathcal{I}$ - Uniform Convergence is stronger than the notion of $\mathcal{I}$- Quasinormal convergence which is again stronger than the notion of $\mathcal{I}$- pointwise convergence. Examples have been constructed in this respect.

Theorem 1

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The following are equivalent

(i) $\operatorname{cof}(\mathcal{I}) = \kappa$.

(ii) *For any set X and for any sequence of real functions, if $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X, then there exist sets X_ξ, $\xi < \kappa$ such that $X = \bigcup_{\xi<\kappa} X_\xi$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_ξ.*

(iii) *For any set $X \subseteq \mathcal{P}(\omega)$ and for any sequence of real functions, if $f_n \xrightarrow{\mathcal{I}\text{QN}} f$ on X, then there exit sets X_ξ, $\xi < \kappa$ such that $X = \bigcup_{\xi<\kappa} X_\xi$ and $f_n \xrightarrow{\mathcal{I}\text{-u}} f$ on each X_ξ.*

Moreover, if X is a topological space and f_n, $n \in \omega$ are continuous, then in both cases we can assume that the sets X_ξ are closed.

Theorem 2

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The following are equivalent:

(i) *The set C is a pseudounion of the ideal $\mathcal{I}$.*

(ii) *For every set X and every $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ on X with the control $\langle \varepsilon_n : n \in \omega \rangle$, there exist sets X_k, $k \in \omega$ such that $X = \bigcup_k X_k$ and $f_n \xrightarrow{\langle C \rangle^*\text{-u}} f$ with same control $\langle \varepsilon_n : n \in \omega \rangle$ on each X_k.*

(iii) *For every set X and every $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} f$ on X with the control $\langle \varepsilon_n : n \in \omega \rangle$, there exists a cardinal (may be finite) κ and there exist sets X_ξ, $\xi < \kappa$ such that $X = \bigcup_{\xi<\kappa} X_\xi$ and $f_n \xrightarrow{\langle C \rangle^*\text{-u}} f$ with same control $\langle \varepsilon_n : n \in \omega \rangle$ on each X_ξ.*

Lemma 1

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Assume that $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal with a pseudounion C. Let $A = \omega \setminus C$. Then

a) For any sequence $\langle x_n : n \in \omega \rangle$ of reals, if $x_n \xrightarrow{\mathcal{I}} 0$ then $x_{e_A(n)} \to 0$.

b) For any sequence $\langle f_n : n \in \omega \rangle$ of real functions defined on X, if $f_n \xrightarrow{\mathcal{I}} 0$ on X, then $f_{e_A(n)} \to 0$ on X.

c) For any sequence $\langle f_n : n \in \omega \rangle$ of real functions defined on X, if $f_n \xrightarrow{\mathcal{I}\mathrm{QN}} 0$ on X, then $f_{e_A(n)} \xrightarrow{\mathrm{QN}} 0$ on X.

- ***$(\mathcal{I},\mathcal{J})$QN-space:*** A topological space X is an $(\mathcal{I},\mathcal{J})$QN-**space** if for any sequence $\langle f_n : n \in \omega \rangle$ of continuous real functions $\mathcal{I}$-converging to 0 on X, we have $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.

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- ***$(\mathcal{I},\mathcal{J})$wQN-space:*** A topological space X is an $(\mathcal{I},\mathcal{J})$wQN-**space** if for any sequence $\langle f_n : n \in \omega \rangle$ of continuous real functions $\mathcal{I}$-converging to zero on X, there exists a sequence $\langle m_n : n \in \omega \rangle$ such that $f_{m_n} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.

- ***($\mathcal{I}$,$\mathcal{J}$)QN-space:*** A topological space X is an ($\mathcal{I}$,$\mathcal{J}$)QN-**space** if for any sequence $\langle f_n : n \in \omega \rangle$ of continuous real functions $\mathcal{I}$-converging to 0 on X, we have $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.
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  • We can take $m_n \xrightarrow{\mathcal{J}} \infty$. Indeed, instead of $\langle f_n : n \in \omega \rangle$, consider the sequence $\langle |f_n| + 2^{-n} : n \in \omega \rangle$. Then for any $a > 0$ and any $x \in X$ we have

$$\{n : m_n \leq a\} \subseteq \{n : |f_{m_n}(x)| + 2^{-m_n} \geq 2^{-a}\} \in \mathcal{J}.$$

- ***$(\mathcal{I},\mathcal{J})$QN-space:*** A topological space X is an $(\mathcal{I},\mathcal{J})$QN-**space** if for any sequence $\langle f_n : n \in \omega\rangle$ of continuous real functions $\mathcal{I}$-converging to 0 on X, we have $f_n \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.
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$$\{n : m_n \leq a\} \subseteq \{n : |f_{m_n}(x)| + 2^{-m_n} \geq 2^{-a}\} \in \mathcal{J}.$$

  • If the sequence $\langle f_n : n \in \omega\rangle$ is decreasing we obtain the notions of an $(\mathcal{I},\mathcal{J})$mQN-space and an $(\mathcal{I},\mathcal{J})$wmQN-space.

Theorem 3

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Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.

a) *If $\mathcal{I}$ has a pseudounion, then every $\mathcal{J}\mathrm{wQN}$-space is an $(\mathcal{I},\mathcal{J})\mathrm{wQN}$-space.*

b) *If $\mathcal{J}$ has a pseudounion, then every $\mathcal{J}\mathrm{QN}$-space is a QN-space and every $(\mathcal{I},\mathcal{J})\mathrm{wQN}$-space is an $(\mathcal{I},\mathrm{Fin})\mathrm{wQN}$-space.*

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• Similar results hold true for $(\mathcal{I},\mathcal{J})$mQN-spaces and $(\mathcal{I},\mathcal{J})$wmQN-spaces.

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- Similar results hold true for $(\mathcal{I},\mathcal{J})$mQN-spaces and $(\mathcal{I},\mathcal{J})$wmQN-spaces.

Corollary

If $\mathcal{I} \subseteq \mathcal{J}$ and the ideal $\mathcal{J}$ has a pseudounion, then every $(\mathcal{I},\mathcal{J})\mathrm{QN}$-space is a QN-space.

Problem

For which ideal $\mathcal{I}$ not containing an isomorphic copy of $\mathrm{Fin}\times\mathrm{Fin}$ do we have $\mathcal{I}\mathrm{QN} \neq \mathrm{QN}$? Similarly for $\mathcal{I}\mathrm{QN}$-, $\mathcal{I}\mathrm{wQN}$-, $\mathcal{I}\mathrm{mQN}$- and $\mathcal{I}\mathrm{wmQN}$-spaces.

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• J. Šupina has very recently showed that assuming $\mathfrak{p} = \mathfrak{c}$, for a γ-space X which is not a QN-space (the existence of such space was proved by Bukovski et al) there exists a tall ideal $\mathcal{I}$, not containing an isomorphic copy of Fin$\times$Fin, such that X is an $\mathcal{I}$QN-space. We can even assume that $\mathcal{I}$ is a maximal ideal. Anyway, that is only very partial answer to our Problem 5.

- (α_1)***:*** We recall that a topological space Y has the **Arkhangel'skiĭ's property** (α_1) if

  (α_1) for any $y \in Y$ and any sequence $\langle\langle y_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of sequences such that $\lim_{m\to\infty} y_{n,m} = y$ for each n, there exists a sequence $\langle z_m : m \in \omega\rangle$ such that $\lim_{m\to\infty} z_m = y$ and $\{y_{n,m} : m \in \omega\} \subseteq^* \{z_m : m \in \omega\}$ for each n,

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- (α_4)***:*** Y has the **Arkhangel'skiĭ's property** (α_4) if

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For ideals $\mathcal{I}$ and $\mathcal{J}$ we can modify the properties (α_1) and (α_4) for the space $C_p(X)$ (or any space of real functions):

($\mathcal{I}, \mathcal{J}$-α_1) *If a sequence* $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of sequences of continuous real functions is such that* $f_{n,m} \xrightarrow{\mathcal{I}} 0$ *for each n, then there exists a sequence* $\langle B_n : n \in \omega\rangle \subseteq \mathcal{J}$, $\bigcup_{n\in\omega} B_n = \omega$, *such that*

$$(\forall\varepsilon > 0)(\forall x \in X)(\exists A \in \mathcal{J})(\forall n, m)(m \notin A \cup B_n \to |f_{n,m}(x)| < \varepsilon). \tag{1}$$

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$$(\forall\varepsilon > 0)(\forall x \in X)(\exists A \in \mathcal{J})(\forall n, m)(m \notin A \cup B_n \to |f_{n,m}(x)| < \varepsilon). \tag{1}$$

and

> $(\mathcal{I},\mathcal{J}$-$\alpha_4)$ *If a sequence* $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of sequences of continuous real functions is such that* $f_{n,m} \xrightarrow{\mathcal{I}} 0$ *for each* n*, then there exists a sequence* $\langle m_n : n \in \omega\rangle$ *such that* $f_{n,m_n} \xrightarrow{\mathcal{J}} 0$.

Theorem 4 and 5

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If X is a topological space then the following are equivalent:

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Theorem

For any topological space X and any ideal $\mathcal{I}$, the following are equivalent.

(i) *X is an $(\mathcal{I},\mathcal{J})\mathrm{QN}$-space.*

(ii) *$C_p(X)$ possesses the property $(\mathcal{I}, \mathcal{J}\text{-}\alpha_1)$.*

We can also introduce the ideal convergence modifications of properties (α_0) and $(\alpha_0)^*$ for the space $C_p(X)$, which were introduced by Bukovski and Hales (2007):

$(\mathcal{I}, \mathcal{J}\text{-}\alpha_0)$ *If a sequence $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of sequences of continuous real functions is such that $f_{n,m} \xrightarrow{\mathcal{I}} 0$ for each n, then there exists a $\mathcal{J}$-diverging to ∞ sequence $\langle n_m : m \in \omega\rangle$ such that $f_{n_m,m} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.*

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and

($\mathcal{I},\mathcal{J}$-α_0^*) *If a sequence* $\langle f_n : n \in \omega\rangle$ *pointwise converges to* 0 *and a sequence* $\langle\langle f_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ *of sequences of continuous real functions is such that* $f_{n,m} \xrightarrow{\mathcal{I}} f_n$ *for each n, then there exists a* $\mathcal{J}$*-diverging to* ∞ *sequence* $\langle n_m : m \in \omega\rangle$ *such that* $f_{n_m,m} \xrightarrow{\mathcal{J}\mathrm{QN}} 0$.

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Theorem 5

Theorem

For a topological space X the following are equivalent.

(i) *X is an $(\mathcal{I},\mathcal{J})\mathrm{QN}$-space.*

(ii) *$C_p(X)$ has the property $(\mathcal{I},\mathcal{J}\text{-}\alpha_0)$.*

(iii) *$C_p(X)$ has the property $(\mathcal{I},\mathcal{J}\text{-}\alpha_0^*)$.*

- ***$\mathcal{I}$-γ-cover:*** Let $\mathcal{I}$ be an ideal. A sequence $\langle U_n : n \in \omega \rangle$ of subsets of a topological space X is said to be an **$\mathcal{I}$-γ-cover**, if for every n, $U_n \neq X$, and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\}$ belongs to $\mathcal{I}$.

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  • We shall identify a countable γ-cover with a Fin-γ-cover. One can easily observe that in this case the enumeration is inessential. The family of all open $\mathcal{I}$-γ-covers of a given topological space X will be denoted by $\mathcal{I}$-Γ.

- ***$\mathcal{I}$-γ-cover:*** Let $\mathcal{I}$ be an ideal. A sequence $\langle U_n : n \in \omega \rangle$ of subsets of a topological space X is said to be an **$\mathcal{I}$-γ-cover**, if for every n, $U_n \neq X$, and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\}$ belongs to $\mathcal{I}$.
  • We shall identify a countable γ-cover with a Fin-γ-cover. One can easily observe that in this case the enumeration is inessential. The family of all open $\mathcal{I}$-γ-covers of a given topological space X will be denoted by $\mathcal{I}$-Γ.
  • A cover $\langle V_n : n \in \omega \rangle$ is called a **refinement of the cover** $\langle U_n : n \in \omega \rangle$ if $V_n \subseteq U_n$ for each $n \in \omega$. An $\mathcal{I}$-γ-cover $\langle U_n : n \in \omega \rangle$ is **shrinkable** if there exists a closed $\mathcal{I}$-γ-cover that is a refinement of $\langle U_n : n \in \omega \rangle$. We denote by $\mathcal{I}$-Γ^{sh} the family of all open shrinkable $\mathcal{I}$-γ-covers.

• If $\langle U_n : n \in \omega \rangle$ and $\langle V_n : n \in \omega \rangle$ are $\mathcal{I}$-γ-covers, then $\langle U_n \cap V_n : n \in \omega \rangle$ is an $\mathcal{I}$-γ-cover. If $\langle U_n : n \in \omega \rangle$ and $\langle V_n : n \in \omega \rangle$ are shrinkable $\mathcal{I}$-γ-covers, then $\langle U_n \cap V_n : n \in \omega \rangle$ is a shrinkable $\mathcal{I}$-γ-cover. Finally, if $\langle U_n : n \in \omega \rangle$ is an $\mathcal{I}$-γ-cover and $U_n \subseteq V_n$ for each n, then $\langle V_n : n \in \omega \rangle$ is an $\mathcal{I}$-γ-cover.

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• For two families $\mathcal{A}$, $\mathcal{B}$ of sequences of subsets of X, we introduce similarly as M. Scheepers did, the property $S_1(\mathcal{A},\mathcal{B})$ as follows: for every sequence $\langle\langle U_{n,m} : m \in \omega \rangle : n \in \omega \rangle$ of sequences from $\mathcal{A}$, there exists a sequence $\langle m_n : n \in \omega \rangle$ of natural numbers such that $\langle U_{n,m_n} : n \in \omega \rangle \in \mathcal{B}$. If a topological space X possesses the property $S_1(\mathcal{A},\mathcal{B})$ we shall say that X is an $S_1(\mathcal{A},\mathcal{B})$ -space.

• A topological space X (or a subset with the subspace topology) with the property $S_1(\Omega, \Gamma)$ is a γ-space, where Ω is the family of all ω-covers [4] of X. The basic results concerning the existence of γ-spaces were proved by F. Galvin and A.W. Miller.

[4] An open cover $\mathcal{A}$ of X is an ω cover, if for every finite $F \subseteq X$, there exists a set $U \in \mathcal{A}$ such that $F \subseteq U$.

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• As above, one can easily show that if X is an $\mathsf{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space, then for every sequence $\langle\langle U_{n,m} : m \in \omega\rangle : n \in \omega\rangle$ of sequences of $\mathcal{I}$-γ-covers there exists a sequence $\langle m_n : n \in \omega\rangle$ of natural numbers such that $m_n \xrightarrow{\mathcal{J}} \infty$ and $\langle U_{n,m_n} : n \in \omega\rangle$ is a $\mathcal{J}$-γ-cover.

[4] An open cover $\mathcal{A}$ of X is an ω cover, if for every finite $F \subseteq X$, there exists a set $U \in \mathcal{A}$ such that $F \subseteq U$.

Theorem 6

Bukovski and hales had found a characterization of wQN-spaces by covers, namely $\mathrm{wQN} \equiv \mathrm{S}_1(\Gamma^{sh}, \Gamma)$. We can show similar result.

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Theorem

If X is a normal topological space, then the following are equivalent:

(i) *X is an $(\mathcal{I}, \mathrm{s}\mathcal{J})\mathrm{wQN}$-space,*

(ii) *X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{sh}, \mathcal{J}\text{-}\Gamma)$-space.*

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THANK YOU FOR YOUR ATTENTION