Separable determination in Asplund spaces

Marek Cúth

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References

M. Cúth, *Separable determination in Asplund spaces*, preprint avaiable at http://www.karlin.mff.cuni.cz/kma-preprints/

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   - Tingley's problem and its relation to the (generalized) lushness
2. Separable determination in Asplund spaces
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Separable determination in general

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In order to join finitely arguments together: the family $\mathcal{F}$ is large.

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Equivalently: Is the mapping $F(x) := \begin{cases} 0 & x = 0 \\ \|x\| f(\frac{x}{\|x\|}) & x \neq 0 \end{cases}$ linear?

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Equivalently: Is it true that

$$\forall \lambda > 0 \; \forall x, y \in S_X : \quad \|f(x) - \lambda f(y)\| = \|x - \lambda y\|?$$

equivalently: instead of "$\|f(x) - \lambda f(y)\| = \|x - \lambda y\|$" it is enough to have "$\|f(x) - \lambda f(y)\| \geq \|x - \lambda y\|$"

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Answer YES if:

- Y arbitrary, X is finite-dimensional and polyhedral
- Y arbitrary, $X \in \{\ell_p, L^p(\mu), \text{Tsirelson space}\}$ $(p \in [1, \infty])$
- Y arbitrary, X is lush (e.g. $X \in \{L^1(\mu), \mathcal{C}(K)\}$)

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Let Z be a Banach space. If the **answer is YES** for Y arbitrary and $X = Z$, we say Z has the **Mazur-Ulam property**.

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Definition

A Banach space X is called *generalized lush* (GL) if for every $x \in S_X$ and every $\varepsilon > 0$ there is $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and, for every $y \in S_X$,

$$\operatorname{dist}(y, S(x^*, \varepsilon)) + \operatorname{dist}(y, -S(x^*, \varepsilon)) < 2 + \varepsilon.$$

Suitable models

Our approach: $\mathcal{F}$ consists of spaces $\overline{X \cap M}$, where M is a suitable model.

Suitable models

Definition

Let $\Phi = \{\varphi_1(x_0, x_1, \ldots, x_{i_1}), \varphi_2(x_0, x_1, \ldots, x_{i_2}), \ldots, \varphi_n(x_0, x_1, \ldots, x_{i_n})\}$ be a finite list of formulas and let Y be a countable set.

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$$\forall j \in \{1, \ldots, n\} \; \forall a_1, \ldots, a_{i_j} \in M :$$

$$\exists x \; \varphi_j(x, a_1, \ldots, a_{i_j}) \Rightarrow \exists x \in M \; \varphi_j(x, a_1, \ldots, a_{i_j}).$$

We write $M \prec (\Phi; Y)$.

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Theorem

Let Φ be a finite list of formulas and Y any countable set. Then there exists a countable set M such that $M \prec (\Phi; Y)$.

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There are a finite list of formulas Φ and a countable set Y such that

$$\mathcal{F} = \{\overline{X \cap M};\ M \prec (\Phi; Y)\}$$

works (i.e. $\forall F \in \mathcal{F}:\ \phi$ holds in X if and only if ϕ holds in F).

Generalized lushness is separably determined in Asplund spaces

Proposition

There are a finite list of formulas Φ and a countable set C such that any $M \prec (\Phi; C)$ satisfies the following:

Let X be a Banach space with $X \in M$. If X is (GL), then $\overline{X \cap M}$ is (GL).

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Theorem

There are a finite list of formulas Φ and a countable set C such that any $M \prec (\Phi; C)$ satisfies the following:

Let X be an Asplund space with $X \in M$. Then

$$X \text{ is (GL)} \iff \overline{X \cap M} \text{ is (GL)}.$$

Key property of Asplund spaces

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$$X^* \cap M \text{ is dense in } (\overline{X \cap M})^*$$

Sketch of the proof

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Lemma

Let X be a Banach space and let $G \subset X^*$ be a dense subset of X^*. Let us assume that there are $x \in S_X$ and $\varepsilon > 0$ such that for every $x^* \in G$ with $x \in S\left(\frac{x^*}{\|x^*\|}, \varepsilon\right)$ there exists $y \in S_X$ such that

$$\mathrm{dist}\left(y, S\left(\frac{x^*}{\|x^*\|}, \varepsilon\right)\right) + \mathrm{dist}\left(y, -S\left(\frac{x^*}{\|x^*\|}, \varepsilon\right)\right) \geq 2 + \varepsilon.$$

Then X is not (GL).

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We try to compare both concepts (rich families and suitable models). This work is still in progress...

The end

Thank you for your attention!