A Cardinality Bound for Hausdorff Spaces

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Twelfth Symposium on General Topology
Prague, July 29, 2016

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- Using convergent open ultrafilters we construct an operator $c : \mathcal{P}(X) \to \mathcal{P}(X)$ with the property that

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- We show $|c(A)| \leq |A|^{\chi(X)}$
- We use a standard closing-off argument

Background

Recall:

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A space X is H-closed if for every open cover $\mathcal{V}$ of X there exists $\mathcal{W} \in [\mathcal{V}]^{<\omega}$ such that $X = \bigcup_{W \in \mathcal{W}} clW$.

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Theorem

A space is H-closed if and only if it is closed in any Hausdorff space in which it is embedded.

In 1982, Dow and Porter proved the following theorems.

Theorem

If X is an H-closed space with a dense set of isolated points then $|X| \leq 2^{\chi(X)}$.

This theorem can be extended to the general Hausdorff setting:

(In fact, the above theorem can be extended further by recent results of Bella and C.).

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If X is a space with a dense set of isolated points then

$$|X| \leq 2^{wL(X)\chi(X)}.$$

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Every H-closed space X can be embedded as the remainder of an H-closed extension Y of a discrete space such that $|X| = |Y|$ and $\chi(X) = \chi(Y)$.

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- Porter gave a simplified approach to the theorem at the top in 1993
- The theorem at the top depends heavily on finiteness and is not known to extend to a general Hausdorff setting

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- Again, this approach seems not to generalize to a general Hausdorff cardinality bound.

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Does there exist a cardinality bound for a Hausdorff space X that generalizes Arhangel'skiĭ's Theorem and the Dow-Porter result?

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Does there exists a property $\mathcal{P}$ of a Hausdorff space that generalizes both Lindelöf and H-closed spaces such that $|X| \leq 2^{\chi(X)}$ for a space X with property $\mathcal{P}$?

The property "almost Lindelöf", a generalization of both H-closed and Lindelöf, would seem to be a natural candidate for the property $\mathcal{P}$.

Definition

For a space X and $A \subseteq X$, the almost Lindelöf degree of A in X, $aL(A, X)$, is the least infinite cardinal κ such that for every open cover $\mathcal{V}$ of A there exists $\mathcal{W} \in [\mathcal{V}]^{\leq\kappa}$ such that $A \subseteq \bigcup_{W \in \mathcal{W}} clW$. The almost Lindelöf degree of X is $aL(X) = aL(X, X)$, and X is almost Lindelöf if $aL(X)$ is countable.

However:

Theorem (Bella/Yaschenko 1998)

If κ is a non-measurable cardinal then there exists an almost-Lindelöf, first-countable Hausdorff space X such that $|X| > \kappa$.

The set $\widehat{U}$ and the invariant $\widehat{L}(X)$

- For a space X, fix an open ultrafilter assignment $f : X \to EX$, where

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Definition

For a non-empty open set $U \subseteq X$, define

$$\widehat{U} = \{x \in X : U \in \mathcal{U}_x\}.$$

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(c) $X \backslash \widehat{U} = \widehat{X \backslash clU}$.

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- The proof relies on the interaction between finiteness in the definition of H-closed and the f.i.p. property of a filter.

Definition

For a space X, define the cardinal invariant $\widehat{L}(X)$ is the least infinite cardinal κ such that for every open cover $\mathcal{V}$ of X there exists $\mathcal{W} \in [\mathcal{V}]^{\leq\kappa}$ such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$.

By the previous Theorem, we see that the property "$\widehat{L}(X) = \aleph_0$" generalizes both H-closed and Lindelöf.

The operator c

Definition

For a space X and $A \subseteq X$, define

$$c(A) = \{x \in X : \widehat{U} \cap A \neq \varnothing \text{ for all } x \in U \in \tau(X)\}.$$

A is c-closed if $A = c(A)$.

Compare with:

$$cl(A) = \{x \in X : U \cap A \neq \varnothing \text{ for all } x \in U \in \tau(X)\}$$

$$cl_\theta(A) = \{x \in X : clU \cap A \neq \varnothing \text{ for all } x \in U \in \tau(X)\},$$

and recall $U \subseteq \widehat{U} \subseteq clU$.

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(f) *If* A *is c-closed then* A *is closed.*
(g) $c(A)$ *is a closed subset of* X.
(h) *If* X *is H-closed then* $c(A)$ *is an H-set.*

Proposition

If X is a space and a C is a c-closed subset of X, then $\widehat{L}(C, X) \leq \widehat{L}(X)$.

I.e., the invariant $\widehat{L}(X)$ is hereditary on c-closed subsets of X.

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  4. otherwise (n,m) is isolated.

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  3. $f(n, m) = \{U \in \tau(\mathbb{U}) : (n, m) \in U\}$ for $(n, m) \in \mathbb{N} \times \mathbb{Z} \backslash (\mathbb{N} \times \{0\})$

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- Since $\{n\} \times \mathbb{N} \subseteq R_n \cup \{\infty\}$, we have $R_n \cup \{\infty\} \in \mathfrak{U}_n$
- Thus $\widehat{R_n \cup \{\infty\}} \cap A \neq \varnothing$ and $\infty \in c(A)$.

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- Since $\{n\} \times \mathbb{N} \subseteq R_n \cup \{\infty\}$, we have $R_n \cup \{\infty\} \in \mathfrak{U}_n$
- Thus $R_n \widehat{\cup \{\infty\}} \cap A \neq \varnothing$ and $\infty \in c(A)$.
- As $S_n \cap (\{n\} \times \mathbb{N} = \varnothing$ for all $n \in \mathbb{N}$, we have $-\infty \notin c(A)$.

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- It is easily seen that $cl_{\mathbb{U}}(A) = A$ and $cl_\theta(A) = A \cup \{\pm\infty\}$. Thus $A \subseteq c(A) \subseteq A \cup \{\pm\infty\}$.
- To see that $\infty \in c(A)$, for $n \in \mathbb{N}$ consider the basic open set $R_n \cup \{\infty\}$ containing ∞.
- Note $\{n\} \times \mathbb{N} \in \mathfrak{U}_n = f(n, 0)$.
- Since $\{n\} \times \mathbb{N} \subseteq R_n \cup \{\infty\}$, we have $R_n \cup \{\infty\} \in \mathfrak{U}_n$
- Thus $\widehat{R_n \cup \{\infty\}} \cap A \neq \varnothing$ and $\infty \in c(A)$.
- As $S_n \cap (\{n\} \times \mathbb{N}) = \varnothing$ for all $n \in \mathbb{N}$, we have $-\infty \notin c(A)$.
- Thus, $c(A) = A \cup \{\infty\}$ and

$$cl(A) \neq c(A) \neq cl_\theta(A).$$

The invariants $aL'(X)$ and $t_c(X)$

Recall:

Definition

For a space X, $aL_c(X)$ is defined as

$$aL_c(X) = \sup\{aL(C, X) : C \text{ is closed}\} + \aleph_0$$

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For a space X, define $aL'(X)$ as

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- $aL'(X) \leq \widehat{L}(X)$ follows from the fact that $\widehat{L}(X)$ is hereditary on c-closed subsets.

Definition

For a space X, the *c-tightness* of X, $t_c(X)$, is defined as the least cardinal κ such that if $x \in c(A)$ for some $x \in X$ and $A \subseteq X$, then there exists $B \in [A]^{\leq\kappa}$ such that $x \in c(B)$.

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Example

Note that $t(\kappa\omega) = \aleph_0$ and $t_c(\kappa\omega) = t(\beta\omega) = \mathfrak{c}$. This shows that $t(\kappa\omega)$ and $t_c(\kappa\omega)$ are not equal.

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Proposition

For any space X,

- $t_c(X) \leq \chi(X)$, *and*
- *if X is regular then $t_c(X) = t(X)$.*

Proposition

For any space X and for all $x \neq y \in X$ there exist open sets U and V such that $x \in U$, $y \in V$, and $\widehat{U} \cap \widehat{V} = \varnothing$.

The above is formally stronger than the usual definition of Hausdorff.

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Proposition

If X is a space and $\psi_c(X) \leq \kappa$, then for all $x \in X$ there exists a family $\mathcal{V}$ of open sets such that $|\mathcal{V}| \leq \kappa$ and

$$\{x\} = \bigcap \mathcal{V} = \bigcap_{V \in \mathcal{V}} clV = \bigcap_{V \in \mathcal{V}} c(\widehat{V}).$$

Proposition

If X is a space and $A \subseteq X$, then

$$|c(A)| \leq |A|^{t_c(X)\psi_c(X)} \leq |A|^{\chi(X)}.$$

Compare the above with:

$$|clA| \leq |A|^{t(X)\psi_c(X)} \leq |A|^{\chi(X)}.$$

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- As $t_c(X) \leq \kappa$, for all $x \in c(A)$ there exists $A(x) \in [A]^{\leq\kappa}$ such that $x \in c(A(x))$.

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- As $t_c(X) \leq \kappa$, for all $x \in c(A)$ there exists $A(x) \in [A]^{\leq\kappa}$ such that $x \in c(A(x))$.
- Define $\phi : c(A) \to \left[[A]^{\leq\kappa}\right]^{\leq\kappa}$ by

$$\phi(x) = \{\widehat{V} \cap A(x) : V \in \mathcal{V}_x\}.$$

Observe that $\phi(x) \in \left[[A]^{\leq\kappa}\right]^{\leq\kappa}$.

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- Fix $x \in c(A)$. It is straightforward to show that $x \in c(\widehat{V} \cap A(x))$ for all $V \in \mathcal{V}_x$.

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- Thus,

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- This shows ϕ is one-to-one and $|c(A)| \leq |A|^{\kappa}$.

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Theorem (Hodel)

Let X be a set, κ be an infinite cardinal, $d : \mathcal{P}(X) \to \mathcal{P}(X)$ an operator on X, and for each $x \in X$ let $\{V(\alpha, x) : \alpha < \kappa\}$ be a collection of subsets of X. Assume the following:

Then $|X| \leq 2^\kappa$.

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- (C) *(cardinality condition) if $A \subseteq X$ with $|A| \leq \kappa$, then $|d(A)| \leq 2^\kappa$;*
- (C-S) *(cover-separation condition) if $H \neq \varnothing$, $d(H) \subseteq H$, and $q \notin H$, then there exists $A \subseteq H$ with $|A| \leq \kappa$ and a function $f : A \to \kappa$ such that $H \subseteq \bigcup_{x \in A} V(f(x), x)$ and $q \notin \bigcup_{x \in A} V(f(x), x)$.*

Then $|X| \leq 2^\kappa$.

Using the operator c in place of the operator d in Hodel's theorem, we obtain:

Main Theorem (C., Porter, 2016)

If X is Hausdorff then

$$|X| \leq 2^{aL'(X)t_c(X)\psi_c(X)} \leq 2^{aL'(X)\chi(X)} \leq 2^{\widehat{L}(X)\chi(X)}.$$

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Compare the above to the following:

Theorem (Bella,Cammaroto)

If X is Hausdorff then $|X| \leq 2^{aL_c(X)t(X)\psi_c(X)}$.

As $aL'(X) \leq \widehat{L}(X)$ and $\widehat{L}(X) = \aleph_0$ for an H-closed space X, it follows that:

Corollary (Dow, Porter 1982)

If X is H-closed then $|X| \leq 2^{\psi_c(X)}$.

We can now identify a property $\mathcal{P}$ of a Hausdorff space X that generalizes both the H-closed and Lindelöf properties such that $|X| \leq 2^{\chi(X)}$ for spaces with property $\mathcal{P}$:

$\mathcal{P} =$ for every open cover $\mathcal{V}$ of X there is $\mathcal{W} \in [\mathcal{V}]^{\leq\omega}$ such that

$$X = \bigcup_{W \in \mathcal{W}} \widehat{W}$$

Questions

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Given relationships between cardinality bounds for general Hausdorff spaces and bounds for homogeneous spaces, we can ask:

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If X is a homogeneous Hausdorff space, is

$$|X| \leq 2^{aL'(X)t_c(X)pct(X)}?$$

Thank you!

C., Jack Porter, *On the Cardinality of Hausdorff and H-closed Spaces*, pre-print.