Universality of group embeddability

Filippo Calderoni

University of Turin
Polytechnic di Turin

27th July 2016

Borel reducibility

In the framework of Borel reducibility, relations are defined over Polish or standard Borel spaces.

Definition

Let E and F be binary relations over X and Y, respectively.

- E **Borel reduces** to F (or $\boldsymbol{E \leq_B F}$) if and only if there is a Borel $f : X \to Y$ such that
$$x_1 \, E \, x_2 \quad \Leftrightarrow \quad f(x_1) \, F \, f(x_2).$$
- E and F are **Borel bi-reducible** (or $\boldsymbol{E \sim_B F}$) if and only if $E \leq_B F$ and $F \leq_B E$.

Comparing equivalence relations

First, the ordering $\leq_B$ can be used to find complete invariants for a given equivalence relation.

Examples

(Gromov) the isometry between compact Polish metric spaces Borel reduces to $=_{\mathbb{R}}$.

(Stone) the homeomorphism between separable compact zero-dimensional Hausdorff spaces Borel reduces to the isomorphism between countable Boolean algebras.

Comparing equivalence relations

First, the ordering $\leq_B$ can be used to find complete invariants for a given equivalence relation.

Examples

- (Gromov) the isometry between compact Polish metric spaces Borel reduces to $=_{\mathbb{R}}$.
- (Stone) the homeomorphism between separable compact zero-dimensional Hausdorff spaces Borel reduces to the isomorphism between countable Boolean algebras.

Comparing equivalence relations

Moreover, the notion of Borel reducibility has been used to get structural results about the class of analytic equivalence relations (quasi-orders)

- defining milestones and see where other equivalence relations fit in the picture,
- dichothomy results (Silver, Harrington-Kechris-Louveau, etc...).

Analytic quasi-orders

Definition

A quasi-order Q defined on X is $\mathbf{\Sigma}^1_1$ (or **analytic**) if it is analytic as a subset of $X \times X$.

Examples

- Fix $\mathcal{L}$ a countable relational language. Any countable $\mathcal{L}$-structure is viewed as an element of $X_{\mathcal{L}} = \prod_{R \in \mathcal{L}} 2^{\mathbb{N}^{a(R)}}$

$$M \sqsubseteq_{\mathcal{L}} N \quad \overset{def}{\iff} \quad \exists h : \mathbb{N} \xrightarrow{1-1} \mathbb{N} \quad h \text{ is an isomorphism from } M \text{ to } N_{|Im(h)}.$$

- If X is a Polish space and G is a Polish monoid such that $a : G \times X \to X$ is a Borel action,

$$x \, R^X_G \, y \quad \overset{def}{\iff} \quad \exists g \in G \, (a(g, x) = y).$$

Analytic quasi-orders

Definition

A quasi-order Q defined on X is $\mathbf{\Sigma}^1_1$ (or **analytic**) if it is analytic as a subset of $X \times X$.

Examples

- Fix $\mathcal{L}$ a countable relational language. Any countable $\mathcal{L}$-structure is viewed as an element of $X_{\mathcal{L}} = \prod_{R \in \mathcal{L}} 2^{\mathbb{N}^{a(R)}}$

$$M \sqsubseteq_{\mathcal{L}} N \quad \overset{def}{\iff} \quad \exists h : \mathbb{N} \xrightarrow{1-1} \mathbb{N} \quad h \text{ is an isomorphism from } M \text{ to } N_{|Im(h)}.$$

- If X is a Polish space and G is a Polish monoid such that $a : G \times X \to X$ is a Borel action,

$$x \; R^X_G \; y \quad \overset{def}{\iff} \quad \exists g \in G \, (a(g, x) = y).$$

Analytic quasi-orders

Definition

A quasi-order Q defined on X is $\mathbf{\Sigma}^1_1$ (or **analytic**) if it is analytic as a subset of $X \times X$.

Examples

- Fix $\mathcal{L}$ a countable relational language. Any countable $\mathcal{L}$-structure is viewed as an element of $X_{\mathcal{L}} = \prod_{R \in \mathcal{L}} 2^{\mathbb{N}^{a(R)}}$

$$M \sqsubseteq_{\mathcal{L}} N \quad \stackrel{def}{\Longleftrightarrow} \quad \exists h : \mathbb{N} \xrightarrow{1-1} \mathbb{N} \quad h \text{ is an isomorphism from } M \text{ to } N_{|Im(h)}.$$

- If X is a Polish space and G is a Polish monoid such that $a : G \times X \to X$ is a Borel action,

$$x \, R^X_G \, y \quad \stackrel{def}{\Longleftrightarrow} \quad \exists g \in G \, (a(g, x) = y).$$

$\mathbf{\Sigma}_1^1$-complete quasi-orders

Definition

A quasi-order Q is **$\mathbf{\Sigma}_1^1$-complete** if and only if Q is $\mathbf{\Sigma}_1^1$ and $P \leq_B Q$, for every $\mathbf{\Sigma}_1^1$ quasi-order P.

Theorem (Louveau-Rosendal 2005)

The embeddability between countable graphs $\sqsubseteq_{\mathrm{Gr}}$ is a $\mathbf{\Sigma}_1^1$-complete quasi-order.

Theorem (Ferenczi-Louveau-Rosendal 2009)

The topological embeddability between Polish groups $\sqsubseteq_{\mathrm{PGp}}$ is a $\mathbf{\Sigma}_1^1$-complete quasi-order.

$\mathbf{\Sigma}^1_1$-complete quasi-orders

Definition

A quasi-order Q is **$\mathbf{\Sigma}^1_1$-complete** if and only if Q is $\mathbf{\Sigma}^1_1$ and $P \leq_B Q$, for every $\mathbf{\Sigma}^1_1$ quasi-order P.

Theorem (Louveau-Rosendal 2005)

The embeddability between countable graphs $\sqsubseteq_{\mathrm{Gr}}$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order.

Theorem (Ferenczi-Louveau-Rosendal 2009)

The topological embeddability between Polish groups $\sqsubseteq_{\mathrm{PGp}}$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order.

$\mathbf{\Sigma}^1_1$-complete quasi-orders

Definition

A quasi-order Q is **$\mathbf{\Sigma}^1_1$-complete** if and only if Q is $\mathbf{\Sigma}^1_1$ and $P \leq_B Q$, for every $\mathbf{\Sigma}^1_1$ quasi-order P.

Theorem (Louveau-Rosendal 2005)

The embeddability between countable graphs $\sqsubseteq_{\mathsf{Gr}}$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order.

Theorem (Ferenczi-Louveau-Rosendal 2009)

The topological embeddability between Polish groups $\sqsubseteq_{\mathsf{PGp}}$ is a $\mathbf{\Sigma}^1_1$-complete quasi-order.

Invariant Universality

Definition

Let Q be a $\boldsymbol{\Sigma}^1_1$ quasi-order and E a $\boldsymbol{\Sigma}^1_1$ equivalence subrelation of Q. We say that the pair (Q, E) is **invariantly universal** if for every $\boldsymbol{\Sigma}^1_1$ quasi-order R there is a Borel $B \subseteq dom(Q)$ such that:

- B is invariant respect to E,
- $Q \restriction B \sim_B R$.

(Q, E) invariantly universal $\Rightarrow$ Q is $\boldsymbol{\Sigma}^1_1$-complete.

$\not\Leftarrow$

Invariant Universality

Definition

Let Q be a $\mathbf{\Sigma}^1_1$ quasi-order and E a $\mathbf{\Sigma}^1_1$ equivalence subrelation of Q. We say that the pair (Q, E) is **invariantly universal** if for every $\mathbf{\Sigma}^1_1$ quasi-order R there is a Borel $B \subseteq dom(Q)$ such that:

- B is invariant respect to E,
- $Q \restriction B \sim_B R$.

$$(Q, E) \text{ invariantly universal} \quad \Rightarrow \quad Q \text{ is } \mathbf{\Sigma}^1_1\text{-complete.}$$
$$\not\Leftarrow$$

Looking for "natural" example

Questions

- Is $(\sqsubseteq_{\mathsf{Gr}}, \cong_{\mathsf{Gr}})$ invariantly universal?
- Is $(\sqsubseteq_{\mathsf{PGp}}, \cong_{\mathsf{PGp}})$ invariantly universal?

Theorem (Friedman-Motto Ros 2011)

There exists a Borel $\mathbb{G} \subseteq X_{Gr}$ such that:

1. *each element of $\mathbb{G}$ is a connected acyclic graph,*
2. *$=_{\mathbb{G}}$ and $\cong_{\mathbb{G}}$ coincide,*
3. *each graph in $\mathbb{G}$ is rigid, i.e. it has no nontrivial automorphism,*
4. *$\sqsubseteq_{\mathbb{G}}$, the embeddability between countable graphs restricted to $\mathbb{G}$, is a complete $\boldsymbol{\Sigma}^1_1$ quasi-orders.*

Theorem (Camerlo-Marcone-Motto Ros 2013)

$(\sqsubseteq_{Gr}, \cong_{Gr})$ is invariantly universal.

Corollary

For every $\boldsymbol{\Sigma}^1_1$ quasi-order Q there exists a $\mathcal{L}_{\omega_1\omega}$-formula φ in the language of graphs such that $Q \sim_B \sqsubseteq_{Gr} \restriction Mod_\varphi$.

Theorem (Friedman-Motto Ros 2011)

There exists a Borel $\mathbb{G} \subseteq X_{Gr}$ such that:

1. *each element of $\mathbb{G}$ is a connected acyclic graph,*
2. *$=_{\mathbb{G}}$ and $\cong_{\mathbb{G}}$ coincide,*
3. *each graph in $\mathbb{G}$ is rigid, i.e. it has no nontrivial automorphism,*
4. *$\sqsubseteq_{\mathbb{G}}$, the embeddability between countable graphs restricted to $\mathbb{G}$, is a complete $\boldsymbol{\Sigma}_1^1$ quasi-orders.*

Theorem (Camerlo-Marcone-Motto Ros 2013)

$(\sqsubseteq_{Gr}, \cong_{Gr})$ is invariantly universal.

Corollary

For every $\boldsymbol{\Sigma}_1^1$ quasi-order Q there exists a $\mathcal{L}_{\omega_1\omega}$-formula φ in the language of graphs such that $Q \sim_B \sqsubseteq_{Gr} \restriction Mod_{\varphi}$.

The only known technique

Theorem (Camerlo-Marcone-Motto Ros 2013)

Let Q be a $\mathbf{\Sigma}^1_1$ quasi-order on X and $E \subseteq Q$ a $\mathbf{\Sigma}^1_1$ equivalence relation. (Q, E) is invariantly universal provided that there is a Borel $f : \mathbb{G} \to X$ such that:

- $\sqsubseteq_{\mathbb{G}} \leq_B Q$ *and* $=_{\mathbb{G}} \leq_B E$ *via* f,
- *there is a reduction* $g : E \leq_B E^Y_H$, *for some standard Borel* H*-space* Y,
- *the map*

$$\begin{aligned} \mathbb{G} &\longrightarrow F(H) \\ T &\longmapsto \mathrm{Stab}(g \circ f(T)) \quad \textit{is Borel.} \end{aligned}$$

The only known technique

Theorem (Camerlo-Marcone-Motto Ros 2013)

Let Q be a $\mathbf{\Sigma}^1_1$ quasi-order on X and $E \subseteq Q$ a $\mathbf{\Sigma}^1_1$ equivalence relation. (Q, E) is invariantly universal provided that there is a Borel $f : \mathbb{G} \to X$ such that:

- $\sqsubseteq_{\mathbb{G}} \leq_B Q$ and $=_{\mathbb{G}} \leq_B E$ via f,
- *there is a reduction* $g : E \leq_B E^Y_H$*, for some standard Borel* H*-space* Y*,*
- *the map*

$$\begin{aligned} \mathbb{G} &\longrightarrow F(H) \\ T &\longmapsto \mathrm{Stab}(g \circ f(T)) \quad \text{is Borel.} \end{aligned}$$

The only known technique

Theorem (Camerlo-Marcone-Motto Ros 2013)

Let Q be a $\mathbf{\Sigma}^1_1$ quasi-order on X and $E \subseteq Q$ a $\mathbf{\Sigma}^1_1$ equivalence relation. (Q, E) is invariantly universal provided that there is a Borel $f : \mathbb{G} \to X$ such that:

- $\sqsubseteq_{\mathbb{G}} \leq_B Q$ *and* $=_{\mathbb{G}} \leq_B E$ *via* f,
- *there is a reduction* $g : E \leq_B E^Y_H$, *for some standard Borel* H*-space* Y,
- *the map*

$$\begin{aligned} \mathbb{G} &\longrightarrow F(H) \\ T &\longmapsto \mathrm{Stab}(g \circ f(T)) \quad \textit{is Borel.} \end{aligned}$$

Topological embeddability on Polish groups

Questions

- Is $(\sqsubseteq_{\mathsf{Gr}}, \cong_{\mathsf{Gr}})$ invariantly universal ?
- Is $(\sqsubseteq_{\mathsf{PGp}}, \cong_{\mathsf{PGp}})$ invariantly universal ?

Theorem (Ferenczi-Louveau-Rosendal 2009)

$\cong_{\mathsf{PGp}}$ is a $\boldsymbol{\Sigma}^1_1$-complete equivalence relation.

Topological embeddability on Polish groups

Questions

- Is $(\sqsubseteq_{\mathsf{Gr}}, \cong_{\mathsf{Gr}})$ invariantly universal
- Is $(\sqsubseteq_{\mathsf{PGp}}, \cong_{\mathsf{PGp}})$ invariantly universal ?

Theorem (Ferenczi-Louveau-Rosendal 2009)

$\cong_{\mathsf{PGp}}$ *is a* $\mathbf{\Sigma}^1_1$*-complete equivalence relation.*

Topological embeddability on Polish groups

Questions

- Is $(\sqsubseteq_{\mathrm{Gr}}, \cong_{\mathrm{Gr}})$ invariantly universal
- Is $(\sqsubseteq_{\mathrm{PGp}}, \cong_{\mathrm{PGp}})$ invariantly universal ?

Theorem (Ferenczi-Louveau-Rosendal 2009)

$\cong_{\mathrm{PGp}}$ is a $\mathbf{\Sigma}^1_1$-complete equivalence relation.

Topological embeddability on Polish groups

Questions

- Is $(\sqsubseteq_{\mathrm{Gr}}, \cong_{\mathrm{Gr}})$ invariantly universal
- Is $(\sqsubseteq_{\mathrm{PGp}}, \cong_{\mathrm{PGp}})$ invariantly universal ?

Theorem (Ferenczi-Louveau-Rosendal 2009)

$\cong_{\mathrm{PGp}}$ is a $\mathbf{\Sigma}^1_1$-complete equivalence relation.

⚠ It is NOT possible to reduce $\cong_{PGp}$ to any Borel group action because $\cong_{PGp}$ is $\mathbf{\Sigma}^1_1$-complete.

Embeddability between countable groups

Theorem (Williams 2014)

The embeddability between countable groups $\sqsubseteq_{\mathsf{Gp}}$ is a $\boldsymbol{\Sigma}^1_1$-complete quasi-order.

Theorem (C.-Motto Ros)

$(\sqsubseteq_{\mathsf{Gp}}, \cong_{\mathsf{Gp}})$ *is invariantly universal.*

Embeddability between countable groups

Theorem (Williams 2014)

The embeddability between countable groups $\sqsubseteq_{\mathsf{Gp}}$ *is a* $\mathbf{\Sigma}^1_1$*-complete quasi-order.*

Theorem (C.-Motto Ros)

$(\sqsubseteq_{\mathsf{Gp}}, \cong_{\mathsf{Gp}})$ *is invariantly universal.*

Embeddability between countable groups

Theorem (C.-Motto Ros)

$(\sqsubseteq_{Gp}, \cong_{Gp})$ *is invariantly universal.*

Proof *(sketch)*

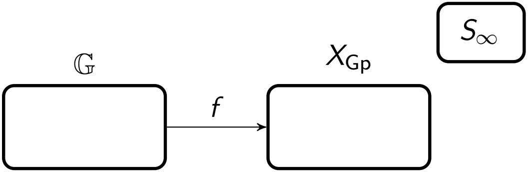
Williams defined a Borel function

$$\begin{aligned} f : X_{Gr} &\longrightarrow X_{Gp} \\ T &\longmapsto G_T. \end{aligned}$$

Every G_T satisfies some small cancellation properties, which are used to prove that f reduces $\sqsubseteq_{Gr}$ to $\sqsubseteq_{\mathrm{Gp}}$.

Moreover, $=_{\mathbb{G}} \leq_B \cong_{\mathrm{Gp}}$ via f.

Embeddability between countable groups

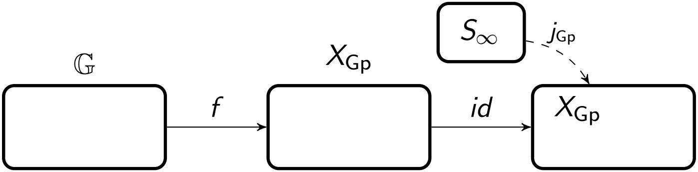


Let S_∞ be the Polish group of all permutations of $\mathbb{N}$.

The logic action of S_∞ on X_{Gp} is continuous and $\cong_{\mathsf{Gp}}$ coincides with $E_{S_\infty}^{X_{\mathsf{Gp}}}$.

$$\mathrm{Stab}(f(T)) = \{h \in S_\infty : j_{\mathsf{Gp}}(h, f(T)) = f(T)\} = \\ = Aut(G_T).$$

Embeddability between countable groups

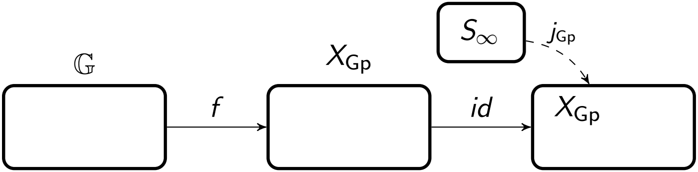


Let S_∞ be the Polish group of all permutations of $\mathbb{N}$.

The logic action of S_∞ on X_{Gp} is continuous and $\cong_{\mathsf{Gp}}$ coincides with $E_{S_\infty}^{X_{\mathsf{Gp}}}$.

$$\mathrm{Stab}(f(T)) = \{h \in S_\infty : j_{\mathsf{Gp}}(h, f(T)) = f(T)\} = \\ = Aut(G_T).$$

Embeddability between countable groups

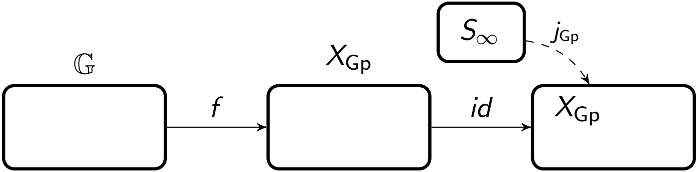


Let S_∞ be the Polish group of all permutations of $\mathbb{N}$.

The logic action of S_∞ on X_{Gp} is continuous and $\cong_{\mathsf{Gp}}$ coincides with $E_{S_\infty}^{X_{\mathsf{Gp}}}$.

$$\begin{aligned}\mathrm{Stab}(f(T)) =& \{h \in S_\infty : j_{\mathsf{Gp}}(h, f(T)) = f(T)\} = \\ =& Aut(G_T).\end{aligned}$$

Embeddability between countable groups



Lemma

Let $B \subseteq X_{\mathsf{Gr}}$ be Borel. If the map $B \to F(S_\infty), T \mapsto Aut(T)$ is Borel, then so is the map $B \to F(S_\infty), T \mapsto Aut(G_T)$.

Apply the Lemma with $B = \mathbb{G}$ and recall that every $T \in \mathbb{G}$ is rigid. $\square$

Topological embeddability on Polish groups

Theorem (C.-Motto Ros)

$(\sqsubseteq_{\mathsf{PGp}}, \cong_{\mathsf{PGp}})$ *is invariantly universal.*

By Uspenskij, every Polish group is homeomorphic to a closed subgroup of $\mathrm{Homeo}([0,1]^{\mathbb{N}})$.

Let $X_{PGp} := Subg(\mathrm{Homeo}([0,1]^{\mathbb{N}}))$ with the Effros Borel structure.

Topological embeddability on Polish groups

Theorem (C.-Motto Ros)

$(\sqsubseteq_{\mathsf{PGp}}, \cong_{\mathsf{PGp}})$ *is invariantly universal.*

By Uspenskij, every Polish group is homeomorphic to a closed subgroup of $\mathrm{Homeo}([0,1]^{\mathbb{N}})$.

Let $X_{PGp} := Subg(\mathrm{Homeo}([0,1]^{\mathbb{N}}))$ with the Effros Borel structure.

Topological embeddability on Polish groups

Theorem (C.-Motto Ros)

$(\sqsubseteq_{\mathsf{PGp}}, \cong_{\mathsf{PGp}})$ *is invariantly universal.*

Proof *(sketch)*

By Williams, there exists a Borel function

$$\begin{aligned} X_{Gr} &\longrightarrow X_{Gp} \\ T &\longmapsto G_T \end{aligned}$$

witnessing $\sqsubseteq_{Gr} \leq_B \sqsubseteq_{Gp}$ and $=_{\mathbb{G}} \leq_B \cong_{\mathbb{G}}$.

Topological embeddability on Polish groups

Theorem (C.-Motto Ros)

$(\sqsubseteq_{\mathrm{PGp}}, \cong_{\mathrm{PGp}})$ *is invariantly universal.*

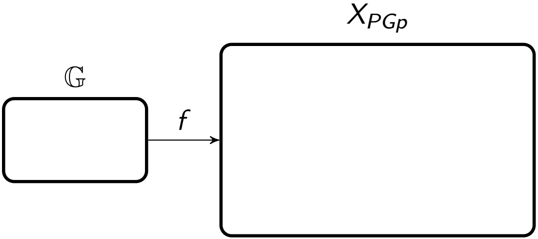
Proof *(sketch)*

By Williams, there exists a Borel function

$$\begin{aligned} X_{Gr} &\longrightarrow X_{Gp} \\ T &\longmapsto G_T \end{aligned}$$

witnessing $\sqsubseteq_{Gr} \leq_B \sqsubseteq_{Gp}$ and $=_{\mathbb{G}} \leq_B \cong_{\mathbb{G}}$.

Topological embeddability on Polish groups

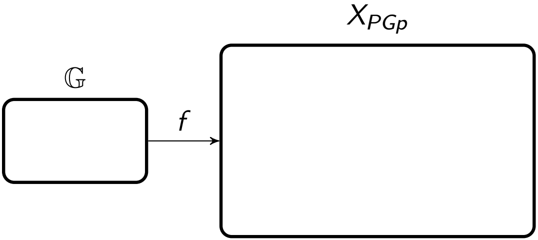


$$T \longmapsto (G_T, \mathcal{P}(G_T)) \rightsquigarrow \text{code of } (G_T, \mathcal{P}(G_T)) \text{ in } X_{PGp}$$

It is NOT possible to reduce $\cong_{PGp}$ to any Borel group action because $\cong_{PGp}$ is $\mathbf{\Sigma}^1_1$-complete.

However, $\operatorname{ran} f \subseteq D = \{F \in X_{PGp} : F \text{ is a discrete group}\}$ which is a $\cong_{PGp}$-invariant Borel subset of X_{PGp}.

Topological embeddability on Polish groups

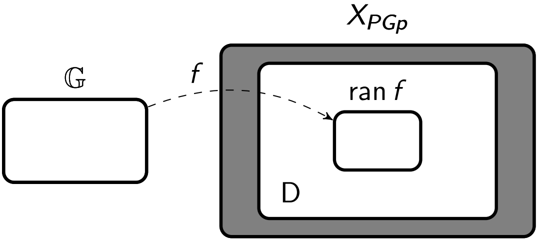


$T \longmapsto (G_T, \mathcal{P}(G_T)) \rightsquigarrow$ code of $(G_T, \mathcal{P}(G_T))$ in X_{PGp}

It is NOT possible to reduce $\cong_{PGp}$ to any Borel group action because $\cong_{PGp}$ is $\mathbf{\Sigma}^1_1$-complete.

However, $\operatorname{ran} f \subseteq D = \{F \in X_{PGp} : F$ is a discrete group$\}$ which is a $\cong_{PGp}$-invariant Borel subset of X_{PGp}.

Topological embeddability on Polish groups

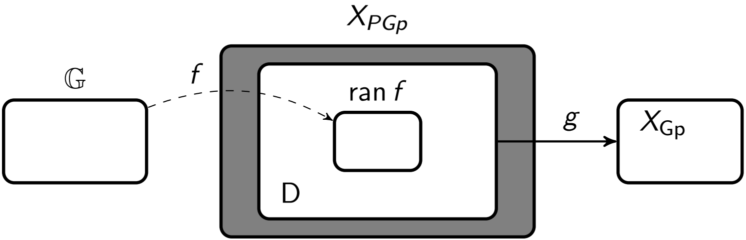


$$T \longmapsto (G_T, \mathcal{P}(G_T)) \rightsquigarrow \text{code of } (G_T, \mathcal{P}(G_T)) \text{ in } X_{PGp}$$

It is NOT possible to reduce $\cong_{PGp}$ to any Borel group action because $\cong_{PGp}$ is $\mathbf{\Sigma}^1_1$-complete.

However, ran $f \subseteq D = \{F \in X_{PGp} : F \text{ is a discrete group}\}$ which is a $\cong_{PGp}$-invariant Borel subset of X_{PGp}.

Topological embeddability on Polish groups



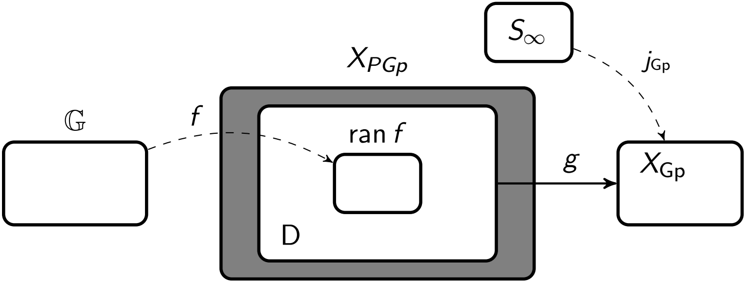
Then, define a Borel $g : D \to X_{\mathrm{Gp}}$ such that
$g : \cong_{PGp} \leq_B \cong_{Gp}$ and $g(f(T)) = G_T$, for every $T \in \mathbb{G}$.

Then, the map

$$\begin{aligned} \mathbb{G} &\longrightarrow F(S_\infty) \\ T &\longmapsto \mathrm{Stab}(g \circ f(T)) = Aut(G_T) \end{aligned}$$

is Borel by the previous theorem. □

Topological embeddability on Polish groups



Then, define a Borel $g : D \to X_{\mathrm{Gp}}$ such that $g : \cong_{PGp} \leq_B \cong_{Gp}$ and $g(f(T)) = G_T$, for every $T \in \mathbb{G}$.

Then, the map

$$\begin{aligned} \mathbb{G} &\longrightarrow F(S_\infty) \\ T &\longmapsto \mathrm{Stab}(g \circ f(T)) = Aut(G_T) \end{aligned}$$

is Borel by the previous theorem. □

Separable metric groups with bounded bi-invariant metric

Fix $K > 0$. Let $\sqsubseteq^i_K$ be the isometric embeddability between separable groups with a bi-invariant metric bounded by K.

Theorem (C.-Motto Ros)

$\sqsubseteq^i_K$ is invariantly universal with respect to the isometrical isomorphism.

Some questions

Question

What about the topological embeddability between Polish ABELIAN groups?

Question

Is the embeddability between ABELIAN groups $\mathbf{\Sigma}^1_1$-complete?

Some questions

Question

What about the topological embeddability between Polish ABELIAN groups?

Question

Is the embeddability between ABELIAN groups $\boldsymbol{\Sigma}^1_1$-complete?

Calderoni F., Motto Ros L.
Universality of group embeddability (in preparation)

Thank you!

Calderoni F., Motto Ros L.
Universality of group embeddability (in preparation)

Thank you!